Estimation Functions for Noisy Signals and their Application to a Phaselocked FM Demodulator

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It is often difficult or even not practicable to compute the spectrum of a nonlinear function whose argument is a noisy signal. Therefore, an approximation method is introduced that substitutes nonlinear functions of noisy signals by so-called estimation functions with elementarily determinable spectra. The application of this technique is demonstrated with the example of a phaselocked FM demodulator. Thereby, new results can be found.

Schätzfunktionen für verrauschte Signale und ihre Anwendung auf einen Phaselock-FM-Demodulator

Die exakte analytische Berechnung der Spektren von nichtlinearen Funktionen verrauschter Signale ist häufig schwierig oder gar nicht durchführbar. Daher wird ein Näherungsverfahren vorgestellt, das nichtlineare Funktionen verrauschter Signale durch sogenannte Schätzfunktionen mit einfach bestimmbaren Spektren ersetzt. Die Anwendung dieses Verfahrens wird an einem Phaselock-FM-Demodulator demonstriert. Dabei werden neue Ergebnisse gewonnen.

1. Introduction

For the solution of numerous technical problems the computation of the power spectrum of nonlinear functions of noisy signals is necessary. Even in more or less elementary cases this leads to major difficulties. An example which drastically illustrates these difficulties is the evaluation of the output spectrum of an ideal FM discriminator. F. M. Gardner comments on this [1]:

> "Difficulty of the problem is perhaps best illustrated by the fact that the exact analysis was not published until some 45 years after the nature of FM was recognized [2], [3], [4]".

Therefore, approximation methods are needed to limit the mathematical effort to an acceptable amount. Such a method will be introduced here and verified by measurements using as example a phaselocked FM demodulator.

2. Estimation Functions for Noisy Signals [5]

Let $v(t)$ be a noisy signal. If $v(t)$ is the input signal of a two-port with the transfer characteristic $y = f(x)$, then the output signal can be described as

$$y(t) = f[v(t)]. \tag{1}$$

$f(x)$ may be of arbitrary nonlinearity. Therefore, the usual expansion of $f(x)$ in powers of x is often of no advantage, especially, if the nonlinearity is essential.

For lack of better solutions, approximations of the output signals are of interest, even if this approximation is only useful for comparatively low noise applications. In this case, an expansion of the output signal in noise quantities is suggested instead of an expansion of the transfer characteristic in x.

To implement this expansion in a mathematically correct way, the noisy signal $v(t)$ must be described in detail. $v(t)$ is a function of a signal $u_S(t)$ and bandlimited noise $u_N(t)$:

$$v(t) = V[u_S(t), u_N(t)]. \tag{2}$$

If necessary, $v(t)$ can be reduced to an elementary superposition of $u_S(t)$ and $u_N(t)$ by introduction of a further (thought) two-port. Therefore, in the following

$$v(t) = u_S(t) + u_N(t) \tag{3}$$

is assumed. According to S. O. Rice [6], $u_N(t)$ can be described as

$$u_N(t) = \sum_{i=-m}^{m} \hat{u}_i \cos(\omega_c t + i\Delta\omega t + \varphi_c + \varphi_i). \tag{4}$$

$\omega_c/2\pi$ is the centre frequency of the noise band and

$$B := (2m+1)\Delta\omega/2\pi \tag{5}$$

is its bandwidth. $\hat{u}_i$ and φ_i are random variables with probability densities

$$p_{ui}(\hat{u}_i) = \frac{\hat{u}_i}{\sigma_i^2} \exp\left(-\frac{\hat{u}_i^2}{2\sigma_i^2}\right) \Theta(\hat{u}_i), \tag{6}$$

$$p_{\varphi i}(\varphi_i) = \frac{1}{2\pi} [\Theta(\varphi_i) - \Theta(\varphi_i - 2\pi)]. \tag{7}$$

Θ is the unit step function; σ_i^2 is proportional to the available power of the noise signal within a band with centre frequency $(\omega_c + i\Delta\omega)/2\pi$ and bandwidth

$$\Delta f := \Delta\omega/2\pi. \tag{8}$$

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If $\langle\ \rangle$ denotes the expected value referring to the $(4m+2)$ variables $\hat{u}_i$ and φ_i, then the mean power of $u_N(t)$ is proportional to

$$\langle u_N^2(t)\rangle = \sum_{i=-m}^{m} \sigma_i^2 . \quad (9)$$

For white noise, the following holds:

$$\sigma_i^2 = 4RkT\Delta f = 4RN_0\Delta f , \quad (10)$$

$$\langle u_N^2(t)\rangle = 4RkTB = 4RN_0B . \quad (11)$$

The average power of the useful signal is obtained by computing the time average of $u_S^2(t)$. Denoting the time average with a bar, the input signal-to-noise ratio of the above mentioned two-port is

$$\frac{P_{S1}}{P_{N1}} = \frac{\overline{u_S^2(t)}}{\langle u_N^2(t)\rangle} = \frac{\overline{u_S^2(t)}}{\sum\limits_{i=-m}^{m} \sigma_i^2} . \quad (12)$$

Large signal-to-noise ratio means

$$\overline{u_S^2(t)} \gg \langle u_N^2(t)\rangle . \quad (13)$$

Then

$$\overline{u_S^2(t)} \gg \langle u_N^2(t)\rangle = \quad (14)$$

$$= \sum_{i=-m}^{m} \sigma_i^2 > \sigma_j^2 = \langle \hat{u}_j^2\rangle/2 \quad \text{for all} \quad |j| \leqq m$$

must hold. Therefore, a Taylor series expansion in $\hat{u}_i$ of the two-port output signal $y = f[v(t)]$ is suggested.

Unfortunately in general

$$u_S^2(t) \gg \hat{u}_j^2 \quad (15)$$

does not always hold, even if (14) holds. This is, for instance, the case for all zero crossings of $u_S(t)$.

Now the fact is that $f(t)$ is only a sample function of a stochastic process. Therefore, it is not the special dependence of $f(t)$ on the $\hat{u}_i$ which is of interest but the average spectrum of $f(t)$ in dependence on the $\langle \hat{u}_i^2\rangle$. The use of functional analytic methods [7] shows that under these suppositions a Taylor series expansion in $\hat{u}_i$ is possible. A mathematical proof is given in [5].

In this context, $f[v(t)]$ may be approximated by a Taylor series of n-th order:

$$f_n(v) := f(v)\big|_{\vec{u}=\vec{c}} + \sum_{k=1}^{n} \frac{1}{k!} \sum_{i1\ldots ik=-m}^{m} (\hat{u}_{i1} - c_{i1}) \cdot \ldots \cdot (\hat{u}_{ik} - c_{ik}) \frac{\partial^k f(v)}{\partial \hat{u}_{i1} \ldots \partial \hat{u}_{ik}} \bigg|_{\vec{u}=\vec{c}} \quad (16)$$

where

$$\vec{u} = (\hat{u}_{-m}, \ldots, \hat{u}_m) \quad (17)$$

is a vector formed by the $\hat{u}_i$ and

$$\vec{c} = (c_{-m}, \ldots, c_m) \quad (18)$$

is the point of expansion. For low noise $\vec{c} = \vec{o}$ is a sensible point of expansion. Then $f_n(v)$ is reduced to

$$f_n(v) = f(v)\big|_{\vec{u}=\vec{o}} + \sum_{i=-m}^{m} \hat{u}_i \frac{\partial f(v)}{\partial \hat{u}_i}\bigg|_{\vec{u}=\vec{o}} + \frac{1}{2} \sum_{i,j=-m}^{m} \hat{u}_i \hat{u}_j \frac{\partial^2 f(v)}{\partial \hat{u}_i \partial \hat{u}_j}\bigg|_{\vec{u}=\vec{o}} + \cdots . \quad (19)$$

It should be noted that $f_n(v)$ is not an expansion in the conventional sense. $f_n(v)$ does not approximate $f(v)$ point by point so that $|f(v) - f_n(v)|$ is small in a certain neighbourhood of $\vec{u} = \vec{c}$. $f_n(v)$ approximates $f(v)$ in a way that $\langle [f(v) - f_n(v)]^2\rangle$ is small in a certain neighbourhood of $\vec{u} = \vec{c}$. Thus, $f_n(v)$ does not approximate the *sample function* $f(v)$ in its values but the *stochastic process* $f(v)$ in its second order moments. Therefore, in the sequel, $f_n(v)$ is called "*n-th order estimation function for* $f(v)$".

The evaluation of a spectrum frequently turns out to be simpler when using estimation functions instead of the original sample functions, because $f_n(v)$ is a polynomial expansion in $\hat{u}_i$. To obtain the spectrum, the autocorrelation function of $f_n(v)$ is determined as

$$R_{fn}(t+\tau, t) := \langle f_n(t+\tau) f_n(t)\rangle . \quad (20)$$

In general, $f_n(v)$ is nonstationary. Thus, R_{fn} does not only depend on τ but also on t. This is, for example, true when dealing with noisy frequency modulated signals. In this case, the average autocorrelation [8] of f_n is determined as

$$\bar{R}_{fn}(\tau) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{fn}(t+\tau, t)\, dt . \quad (21)$$

The average spectrum is then

$$S_{fn}(f) = \int_{-\infty}^{\infty} \bar{R}_{fn}(\tau)\, e^{-j2\pi f\tau}\, d\tau . \quad (22)$$

Frequently, the noise portion of $S_{fn}(f)$ differs only little from a spectrum that could have been obtained from $f_n(v)$ with a marginal condition that enforces the stationarity of the process. This is true for noisy FM signals with periodic modulation [5]. It is, therefore, advantageous to define "*stationary estimation functions*" $f_{sn}(v)$:

$$f_{sn}(v) := f(v)\big|_{\vec{u}=\vec{c}} + [f_n(v) - f(v)\big|_{\vec{u}=\vec{c}}]\big|_\alpha . \quad (23)$$

α is the marginal condition that enforces stationarity.

As in the case of "normal" Taylor series expansions, it depends on the special process f whether the expansion holds only for a small or for a larger neighbourhood of the expansion point.

3. The Ideal FM Discriminator

It is known that under certain circumstances phaselocked FM demodulators work like ideal discriminators. It is, therefore, useful first to examine the ideal FM discriminator to get results for purposes of comparison.

If the useful signal $u_S(t)$ is modulated in frequency, then it can be described as

$$u_S(t) = \hat{u}_c \cos[\omega_c t + \varphi_c + \varphi(t)] . \quad (24)$$

The information to be transmitted is contained in the time derivative of $\varphi(t)$. With eqs. (3) and (4) the noisy signal $v(t)$ is obtained as

$$v(t) = U_1(t)\cos(\omega_c t + \varphi_c) - U_2(t)\sin(\omega_c t + \varphi_c) \quad (25)$$

with

$$U_1(t) = \hat{u}_c \cos\varphi(t) + \sum_{i=-m}^{m} \hat{u}_i \cos(i\Delta\omega t + \varphi_i), \quad (26)$$

$$U_2(t) = \hat{u}_c \sin\varphi(t) + \sum_{i=-m}^{m} \hat{u}_i \sin(i\Delta\omega t + \varphi_i). \quad (27)$$

Then, $v(t)$ can be written as

$$v(t) = R(t)\cos[\omega_c t + \varphi_c + \varphi(t)] \quad (28)$$

where

$$\tan\varphi(t) := \frac{U_2(t)}{U_1(t)} \quad (29)$$

and

$$R^2(t) := U_1^2(t) + U_2^2(t). \quad (30)$$

Similar results were given by S. O. Rice [6]. From the eqs. (26)—(28) it can be concluded that

$$R(t) = \frac{U_1(t)}{\cos\varphi(t)} = \frac{U_2(t)}{\sin\varphi(t)}. \quad (31)$$

In [10] it is shown that in general

$$R(t) \neq \sqrt[+]{U_1^2(t) + U_2^2(t)} \quad (32)$$

and that $R(t)$ is continuously differentiable. This is important for the expansion of $R(t)$ which is necessary when examining the phaselocked FM demodulator.

An ideal FM discriminator produces an output signal (33)

$$u_D(t) = K\dot{\varphi}(t) = K\frac{U_1(t)\dot{U}_2(t) - U_2(t)\dot{U}_1(t)}{R^2(t)}$$

out of a signal $v(t)$ as in eq. (28). $u_D(t)$ shall now be expanded into estimation functions at $\vec{u} = \vec{o}$. The 0-th order estimation function then describes the noise free case:

$$u_{D0}(t) = u_D(t)|_{\vec{u}=\vec{o}} = K\dot{\varphi}(t). \quad (34)$$

($\varphi(t)$ is defined in eq. (24).)

That is the expected result. To obtain stationary estimation functions, the marginal condition $\varphi(t) \equiv 0$ is chosen. In [5] is shown that this condition does not influence the noise spectrum of the demodulated signal essentially within the baseband. Then (35)

$$u_{SD1}(t) = u_D(t)|_{\vec{u}=\vec{o}} + [u_{DN}(t) - u_D(t)|_{\vec{u}=\vec{o}}]|_{\varphi\equiv 0} = \\ = K\dot{\varphi}(t) + \sum_{i=-m}^{m} \hat{u}_i \frac{\partial}{\partial \hat{u}_i}[u_D(t) - K\dot{\varphi}(t)]\Big|_{\substack{\vec{u}=\vec{o}\\ \varphi\equiv 0}}$$

is a first order stationary estimation function. The evaluation of eq. (35) yields (36)

$$u_{SD1}(t) = K\dot{\varphi}(t) + K\sum_{i=-m}^{m} \frac{\hat{u}_i}{\hat{u}_c} i\Delta\omega\cos(i\Delta\omega t + \varphi_i).$$

This is known as the first order approximation that could have been obtained by different methods [9].

A second order stationary estimation function is given by (37)

$$u_{SD2}(t) = u_{SD1}(t) + \tfrac{1}{2}\sum_{i,j=-m}^{m} \hat{u}_i \hat{u}_j \frac{\partial^2 u_D}{\partial \hat{u}_i \partial \hat{u}_j}\Big|_{\substack{\vec{u}=\vec{o}\\ \varphi\equiv 0}}.$$

Then

$$u_{SD2}(t) = K\dot{\varphi}(t) + K\sum_{i=-m}^{m} \frac{\hat{u}_i}{\hat{u}_c} i\Delta\omega\cos(i\Delta\omega t + \varphi_i) - \\ - \frac{K}{2}\sum_{i,j=-m}^{m} \frac{\hat{u}_i\hat{u}_j}{\hat{u}_c^2}(i+j)\cdot \\ \cdot \Delta\omega\cos[(i+j)\Delta\omega t + \varphi_i + \varphi_j] \quad (38)$$

is obtained. To evaluate the spectra of $u_{SD1}(t)$ and $u_{SD2}(t)$, the abbreviations

$$v_1(t) := K\sum_{i=-m}^{m} \frac{\hat{u}_i}{\hat{u}_c} i\Delta\omega\cos(i\Delta\omega t + \varphi_i), \quad (39)$$

$$v_2(t) := -\frac{K}{2}\sum_{,j=-m}^{m} \frac{\hat{u}_i\hat{u}_j}{\hat{u}_c^2}(i+j)\cdot \\ \cdot \Delta\omega\cos[(i+j)\Delta\omega t + \varphi_i + \varphi_j] \quad (40)$$

are introduced.

The individual sum terms in eq. (39) are all uncorrelated. Therefore, the double-sided spectral density $S_{v1}(i\Delta f)$ is given as (41)

$$S_{v1}(i\Delta f)\Delta f = \begin{cases} K^2\left\langle\left[\frac{\hat{u}_i}{\hat{u}_c} i\Delta\omega\cos(i\Delta\omega t + \varphi_i)\right]^2\right\rangle & \text{for } -m \leqq i \leqq m, \\ 0 & \text{for } |i| > m. \end{cases}$$

In case of white noise it follows that (42)

$$S_{v1}(i\Delta f) = \begin{cases} K^2(2\pi i\Delta f)^2/\hat{u}_c^2\cdot 4RN_0 & \text{for } -m \leqq i \leqq m, \\ 0 & \text{for } |i| > m. \end{cases}$$

In the limit $\Delta f \to 0$; $i\Delta f \to f$ (43)

$$S_{v1}(f) = (2\pi f)^2\frac{4RN_0}{\hat{u}_c^2}K^2[\Theta(f+B/2) - \Theta(f-B/2)]$$

is obtained. Θ is the unit step function. Then the one-sided spectral density is (44)

$$S_{Sv1}(f) = (2\pi K)^2\cdot\frac{N_0}{P_{S1}}[\Theta(f) - \Theta(f-B/2)]f^2$$

where $P_{S1} = \hat{u}_c^2/8R$ is the power of the useful signal at the input of the ideal discriminator. Eq. (44) shows the known quadratic behaviour of demodulated low-noise FM signals.

$v_1(t)$ and $v_2(t)$ are uncorrelated. Therefore, the total spectrum of the noise portion of $u_{SD2}(t)$ is given by the superposition of the spectra for $v_1(t)$ and $v_2(t)$. By rearrangement of the series in eq. (40), the sum terms are ordered according to increasing frequencies:

$$v_2(t) = -\frac{K}{2\hat{u}_c^2}\sum_{i=1}^{2m}\sum_{j=0}^{2m-i} i\Delta\omega\cdot \\ \cdot[\hat{u}_{m-j}\hat{u}_{j-m+i}\cos(i\Delta\omega t + \varphi_{m-j} + \varphi_{j-m+i}) - \\ - \hat{u}_{m-j-i}\hat{u}_{j-m}\cos(i\Delta\omega t - \varphi_{j-m} - \varphi_{m-j-i})]. \quad (45)$$

It can be shown that in eq. (45) only terms of equal frequency are correlated. The one-sided spectral density $S_{Sv2}(i\Delta f)$ is then

$$S_{\mathrm{Sv2}}(i\Delta f)\,\Delta f = \begin{cases} \dfrac{K^2}{4\,\hat{u}_\mathrm{c}^4}\,(2\pi i \Delta f)^2 \cdot \\ \cdot \sum\limits_{j=0}^{2m-i} \langle \hat{u}^2_{m-j}\,\hat{u}^2_{j-m+i} + \hat{u}^2_{j-m}\,\hat{u}^2_{m-j-i} \rangle \\ \quad \text{for} \quad 1 \leqq i \leqq 2m, \\ 0 \quad \text{for} \quad i > 2m \quad \text{and} \quad i < 1. \end{cases} \tag{46}$$

In case of white noise at the demodulator input it follows with eq. (5)

$$S_{\mathrm{Sv2}}(i\Delta f) = \begin{cases} (K^2/2)\,(2\pi)^2\,(N_0/P_{\mathrm{S1}})^2 \cdot (i\Delta f)^2\,(B - i\Delta f) \\ \quad \text{for} \quad 1 \leqq i \leqq 2m, \\ 0 \quad \text{for} \quad i > 2m \quad \text{and} \quad i < 1. \end{cases} \tag{47}$$

In the limit $\Delta f \to 0$; $i\Delta f \to f$,

$$S_{\mathrm{Sv2}}(f) = (K^2/2)\,(2\pi)^2\,(N_0/P_{\mathrm{S1}})^2 \cdot \cdot [\Theta(f) - \Theta(f - B)]\, f^2\,(B - f) \tag{48}$$

is obtained.

The spectrum of v_2 has twice the bandwidth of the spectrum of v_1. Furthermore, the spectrum is no longer quadratic. The difference to a quadratic shape is, indeed, not so large for baseband frequencies ($0 \leqq f \leqq B_\mathrm{B}$). Within the baseband, the total second order noise spectrum is

$$S_{\mathrm{S2}}(f) = (2\pi)^2 K^2 (N_0/P_{\mathrm{S1}}) \cdot \cdot f^2 [1 + \tfrac{1}{2}(N_0/P_{\mathrm{S1}})\,(B - f)]. \tag{49}$$

The second order signal-to-noise ratio at the output is

$$P_{\mathrm{S2}}/P_{\mathrm{N2}} = \overline{K^2 \dot{\varphi}^2} \Big/ \left[\int_0^{B_\mathrm{B}} S_{\mathrm{S2}}(f)\,\mathrm{d}f \right]. \tag{50}$$

Assuming

$$\dot{\varphi}(t) =: 2\pi \Delta F \cos(\omega_\mathrm{L} t + \varphi_\mathrm{L}) \tag{51}$$

it follows that

$$\frac{P_{\mathrm{S2}}}{P_{\mathrm{N2}}} = \frac{3}{2}\,\frac{P_{\mathrm{S1}}}{N_0 B_\mathrm{B}} \left(\frac{\Delta F}{B_\mathrm{B}}\right)^2 \frac{1}{1 + \dfrac{1}{2}\dfrac{N_0 B_\mathrm{B}}{P_{\mathrm{S1}}}\left(\dfrac{B}{B_\mathrm{B}} - \dfrac{3}{8}\right)}. \tag{52}$$

This is a very interesting result: at a given signal power P_{S1} and noise spectral density N_0 at the input, the output signal-to-noise ratio decreases with increasing input bandwidth B. This means that even an ideal FM discriminator needs a preceding preselection filter to show optimal noise behaviour. This result will be of importance when comparing it to the results obtained by a phase-locked FM demodulator.

Eq. (52) shows, moreover, that the output signal-to-noise ratio decreases directly proportional with the input signal-to-noise ratio when P_{S1}/N_0 is large and more than directly proportional when P_{S1}/N_0 is low. This proves the existence of an FM detection threshold.

It appears that the mathematical effort to obtain these (approximating) results is small compared to the case of an exact solution. Therefore, it can be expected that the method of estimation functions could be of benefit even for problems that cannot be solved exactly or where the exact solution is too difficult.

4. A First-Order Phase-Lock-Loop FM Demodulator

One typical problem where the exact method is not applicable is a first-order phase-lock-loop FM demodulator. Exact solutions can only be obtained for the distribution of a phase difference between the noisy input signal phase and the locally generated reference signal phase [11]. An exact solution for the output signal spectrum is not yet available.

It shall be shown that the method of estimation functions yields excellent results for low-noise applications. Fig. 1 shows a first-order PLL demodulator. It should be noted that no limiter is provided to precede the loop.

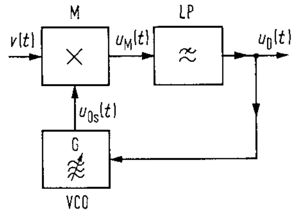

Fig. 1. First-order phase-lock-loop FM demodulator.

The demodulator consists of a multiplying mixer M, a low pass filter LP and a voltage controlled oscillator VCO. With the signal notations of Fig. 1, the circuitry can be described as follows:

$$u_\mathrm{M}(t) = k_\mathrm{M}\, u_{\mathrm{Os}}(t)\, v(t), \tag{53}$$

$$u_{\mathrm{Os}}(t) = \hat{u}_{\mathrm{Os}} \sin[\omega_{\mathrm{Os}} t + k_{\mathrm{Os}} \int^t u_\mathrm{D}(\tau)\,\mathrm{d}\tau], \tag{54}$$

$$u_\mathrm{D}(t) = u_\mathrm{M}(t) * h_{\mathrm{LP}}(t). \tag{55}$$

k_M is a constant of the mixer, k_{Os} is the modulation sensitivity of the oscillator and $\omega_{\mathrm{Os}}/2\pi$ is its centre frequency and $h_{\mathrm{LP}}(t)$ is the impulse response of the low-pass filter.

According to eqs. (24) to (31), the noisy FM signal $v(t)$ can be written as

$$v(t) = R(t) \cos[\omega_\mathrm{c} t + \varphi_\mathrm{c} + \varphi(t)]. \tag{56}$$

Then, with eqs. (53), (54)

$$u_\mathrm{M}(t) = \tfrac{1}{2} k_\mathrm{M} \hat{u}_{\mathrm{Os}} R(t) \cdot \cdot \{-\sin[(\omega_\mathrm{c} - \omega_{\mathrm{Os}})t + \varphi(t) - - k_{\mathrm{Os}} \int^t u_\mathrm{D}(\tau)\,\mathrm{d}\tau + \varphi_\mathrm{c}] + \sin[(\omega_\mathrm{c} + \omega_{\mathrm{Os}})t + + \varphi(t) + k_{\mathrm{Os}} \int^t u_\mathrm{D}(\tau)\,\mathrm{d}\tau + \varphi_\mathrm{c}]\} \tag{57}$$

is obtained. With an appropriate choice of the limit frequency of the lowpass filter LP follows

$$u_\mathrm{D}(t) = -\tfrac{1}{2} k_\mathrm{M} \hat{u}_{\mathrm{Os}} R(t) \cdot \cdot \sin[(\omega_\mathrm{c} - \omega_{\mathrm{Os}})t + \varphi(t) - - k_{\mathrm{Os}} \int^t u_\mathrm{D}(\tau)\,\mathrm{d}\tau + \varphi_\mathrm{c}]. \tag{58}$$

By introduction of the phase difference

$$x(t) := (\omega_\mathrm{c} - \omega_{\mathrm{Os}})t + \varphi(t) - - k_{\mathrm{Os}} \int^t u_\mathrm{D}(\tau)\,\mathrm{d}\tau + \varphi_\mathrm{c} + \pi, \tag{59}$$

eq. (58) is reduced to

$$u_D(t) = \tfrac{1}{2} k_M \hat{u}_{Os} R(t) \sin x(t). \quad (60)$$

Derivating $x(t)$ with respect to the time t and inserting eq. (60) yields

$$\dot{x}(t) + \omega_p \frac{R(t)}{\hat{u}_c} \sin x(t) = \dot{\varphi}(t) + \omega_c - \omega_{Os}. \quad (61)$$

In eq. (61), the abbreviation

$$\omega_p := \tfrac{1}{2} k_M k_{Os} \hat{u}_c \hat{u}_{Os} = 2\pi f_p \quad (62)$$

is used. The simultaneous solution of eqs. (61) and (60) yields the demodulated output signal $u_D(t)$ of a first-order PLL demodulator. It is noteworthy that eqs. (60) and (61) include the time dependent amplitude $R(t)$ of the noisy input signal: The amplitude of the input signal is not assumed to be constant!

A usual way to solve an equation like eq. (61) is the application of the method of Volterra functionals [12]. Then a first-order approximation of eqs. (60) and (61) is

$$\dot{x}_1(t) + \omega_p \frac{R(t)}{\hat{u}_c} x_1(t) = \dot{\varphi}(t) + \omega_c - \omega_{Os}, \quad (63)$$

$$u_D^{(1)}(t) = \frac{\omega_p}{k_{Os}} \frac{R(t)}{\hat{u}_c} x_1(t). \quad (64)$$

The solution of eqs. (63), (64) is

$$u_D^{(1)}(t) = \frac{\omega_p R(t)}{k_{Os}\hat{u}_c} \exp\left[-\frac{\omega_p}{\hat{u}_c} \int^t R(\tau)\,d\tau\right] \cdot \quad (65)$$
$$\cdot \int^t [\dot{\varphi}(\xi) + \omega_c - \omega_{Os}] \exp\left[\frac{\omega_p}{\hat{u}_c} \int^\xi R(\tau)\,d\tau\right] d\xi.$$

This solution can be simplified by use of the method of estimation functions. Therefore, $u_D^{(1)}(t)$ is expanded at $\vec{u} = \vec{o}$. Then, the 0-th order estimation function yields the demodulated signal in the absence of noise:

$$u_{D0}^{(1)}(t) = \frac{\omega_p}{k_{Os}} \exp(-\omega_p t) \cdot \quad (66)$$
$$\cdot \int^t [\dot{\varphi}(\xi) + \omega_c - \omega_{Os}] \exp(\omega_p \xi)\,d\xi.$$

This is the known result [13]: in a first-order approximation and in absence of noise, the PLL demodulator behaves like an ideal FM discriminator, followed by a first-order low-pass filter with limit frequency $f_p = \omega_p/2\pi$.

Stationary estimation functions of higher order can be obtained by the marginal condition $\dot{\varphi}(t) \equiv 0$; $\omega_c - \omega_{Os} = \Delta\Omega$. A first-order estimation function is then $u_{SD1}^{(1)}(t)$ with

$$u_{SD1}^{(1)}(t) - u_{D0}^{(1)}(t) =: v_1(t) =$$
$$= \frac{\omega_p}{k_{Os}} \sum_{k=-m}^{m} \frac{k\Delta\omega - \Delta\Omega}{\omega_p[\omega_p^2 + (k\Delta\omega - \Delta\Omega)^2]} \cdot \quad (67)$$
$$\cdot \{[\omega_p^2 + \Delta\Omega(k\Delta\omega - \Delta\Omega)] \cos(k\Delta\omega t + \varphi_k - \Delta\Omega)$$
$$+ \omega_p(k\Delta\omega - 2\Delta\Omega) \sin(k\Delta\omega t + \varphi_k - \Delta\Omega)\}.$$

Following the methods used in Section 3, the one-sided spectral density of $v_1(t)$ for baseband frequencies is found to be

$$S_{Sv1}(f) = K^2 (2\pi)^2 \frac{N_0}{P_{S1}} \frac{f^2}{1 + f^2/f_p^2} \left[1 + \left(\frac{\Delta F}{f_p}\right)^2\right] \quad (68)$$

whereby

$$K = 1/k_{Os} \quad (69)$$

is the demodulation constant of the PLL demodulator.

Eq. (68) shows that the first-order noise spectrum of the PLL demodulator behaves only in case of $\Delta F = 0$ like that of an ideal FM discriminator followed by a low-pass filter.

This outcome cannot be found by usual methods. It shows that, even with a first-order approximation, the method of estimation functions provides interesting results.

It can be shown by a stationary second-order estimation function that there is a further essential difference to a circuitry, consisting of an ideal demodulator and a low-pass filter. With $\Delta F = 0$ the total second order noise spectrum is (within the baseband)

$$S_{S2}^{(1)}(f) = (2\pi)^2 K^2 \frac{N_0}{P_{S1}} \frac{f^2}{1 + f^2/f_p^2} \cdot \quad (70)$$
$$\cdot \left[1 + \frac{1}{2} \frac{N_0 f_p}{P_{S1}} \left(\arctan\frac{B}{2f_p} + \arctan\frac{B - 2f}{2f_p}\right)\right].$$

Two limiting cases have to be considered. If $B \ll 2f_p$, then it follows that

$$S_{S2}^{(1)}(f) = (2\pi)^2 K^2 \frac{N_0}{P_{S1}} \frac{f^2}{1 + f^2/f_p^2} \cdot \quad (71)$$
$$\cdot \left[1 + \frac{1}{2} \frac{N_0}{P_{S1}} (B - f)\right].$$

In this case, the PLL demodulator works like an ideal demodulator with low-pass filter, provided that there is any circuitry which limits the input bandwidth of the demodulator to B.

If $B \gg 2f_p$, then

$$S_{S2}^{(2)}(f) = (2\pi)^2 K^2 \frac{N_0}{P_{S1}} \frac{f^2}{1 + f^2/f_p^2} \cdot \quad (72)$$
$$\cdot \left[1 + \frac{1}{2} \frac{N_0}{P_{S1}} \pi f_p\right].$$

This is, indeed, a very important difference to a circuitry with ideal demodulator and low-pass filter: eq. (72) shows that there no input bandpass is necessary to keep the value of $S_{S2}^{(1)}(f)$ at a finite value.

The order of magnitude of $\frac{1}{2}(N_0/P_{S1})\pi f_p$ shall be illustrated by an example. FM broadcast systems are using frequency deviations of at most 75 kHz and audio frequencies of up to 53 kHz. The allowed high frequency bandwidth is about $B = 180$ kHz. Choosing the value $f_p = 115$ kHz guarantees that the PLL works within reasonable conditions. Then

$$\frac{1}{2} \frac{N_0}{P_{S1}} \pi f_p \approx \frac{N_0 B}{P_{S1}} = \frac{P_{N1}}{P_{S1}} \quad (73)$$

holds. This means that for input signal-to-noise ratios of more than 6 dB, the first-order and the second-order spectrum differ by less than 1 dB!

This estimation does not take into account effects of nonstationarity (clicks etc.).

The comparison of eqs. (71) and (72) yields that the PLL FM demodulator can be approximated by a circuitry, consisting of a bandpass with centre frequency $\omega_c/2\pi$ and a bandwidth of about πf_p, an ideal FM discriminator, and a first-order low-pass filter with limit frequency f_p. If f_p is comparable to or even higher than the limit frequency of the built-in low-pass filter LP (see Fig. 1), then LP must be taken into account additionally.

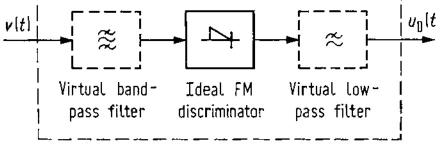

Fig. 2. Equivalent model of a first-order PLL FM demodulator.

This circuitry is shown in Fig. 2. It should be noted that these results were obtained provided that no limiter was used.

In the circuitry of Fig. 2, f_p is an essential parameter. Following eq. (62), f_p is directly proportional to the amplitude $\hat{u}_c$ of the useful signal. Therefore, the (virtual) bandwidths of the PLL demodulator are proportional to $\hat{u}_c$. This "adaptive" behaviour is the object of a research project at the Lehrstuhl für Hoch- und Höchstfrequenztechnik der Universität des Saarlandes [14].

More exact results are obtained, if the approximation by Volterra functionals is implemented for higher orders. Figs. 3 and 4 show the computed

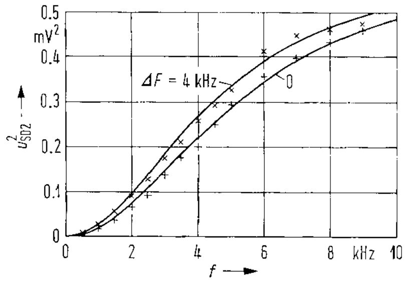

Fig. 3. Computed and measured output signal spectrum of a first-order PLL FM demodulator; $N_0/P_{S1} = (200\ \text{kHz})^{-1}$.

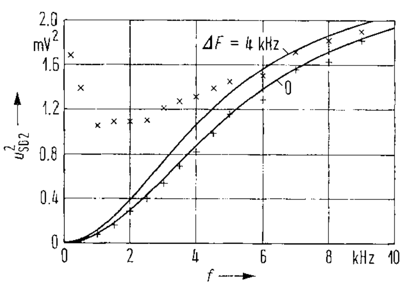

Fig. 4. Computed and measured output signal spectrum of a first-order PLL FM demodulator; $N_0/P_{S1} = (50\ \text{kHz})^{-1}$.

spectrum with third-order Volterra expansion and a stationary first-order estimation function. Stationarity condition was $\omega_{0s} - \omega_c = \Delta F = \text{const.}$

Parameters in the curves of Figs. 3 and 4 are: $f_p = 5.45$ kHz, $f_L = 100$ Hz, measuring bandwidth $B_m = 30$ Hz. The marks ($\times$, $+$) in the figures show measured values.

In Fig. 3, $P_{S1}/N_0 = 200$ kHz was choosen. This corresponds to an input signal-to-noise ratio of about 10 dB when using a system with $\Delta F_{max} = 4$ kHz, $f_{Lmax} = f_p$. It can be seen that measurements and theoretical curves are in excellent accordance.

Fig. 4 shows the results with $P_{S1}/N_0 = 50$ kHz. This corresponds to an input signal-to-noise ratio of about 4 dB. Measurements and theoretical curves are only in good accordance for the stationary case ($\Delta F = 0$). With modulation ($\dot{\varphi} = 2\pi\Delta F \cos\omega_L t$; $\Delta F = 4$ kHz), at low frequencies the measured results differ considerably from the theoretical value. This is due to the fact that there are non-stationary effects (clicks) which cannot be included by stationary estimation functions.

The measurements were made with an input noise bandwidth of $B = 3$ MHz. This input noise bandwidth is, in fact, very large compared to $f_p = 5.45$ kHz. The accordance of measurements and theory (in Fig. 3) gives evidence to the statement that an input bandpass preceding the phaselock demodulator is in principle not necessary, when f_p is choosen to have a suitable value.

5. Summary

With the method of estimation functions, a relatively elementary technique is given to evaluate spectra of nonlinear functions of noisy signals.

The application of this method to a first-order phaselocked FM demodulator shows that theoretical and practical results are in excellent agreement if the suppositions of the approximation are satisfied.

A new result is that the phaselocked FM demodulator behaves like an FM demodulator with a preceding virtual bandpass filter.

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