Have 1/*f* Noise and (Quantum) Thermal Noise the Same Physical Causes?

Michael H.W. Hoffmann

Department of Microwave Techniques, University of Ulm, Albert-Einstein-Allee 41, D-89069 Ulm, Germany

Abstract. Modern models on thermal noise are derived based on the quantum-theoretical fluctuation-dissipation theorem (F-D Theorem). Although 1/*f*-noise is apparently a fluctuation phenomenon, it is not (yet?) covered by the F-D theorem. The reason for that could be that the F-D theorem is subject to limiting assumptions that are not met for important experimental setups. It is expected that a rigorous revision of the F-D theorem will not only explain some discrepancies at the high end of the thermal noise spectrum but could also explain 1/*f*-noise on a very broad basis.

THE PROBLEM

Thermal noise and 1/*f*-noise are phenomena that are well known since decades. It seems to be widely accepted that thermal noise could be explained by fluctuations of quantum states in an “impedance” and that these fluctuations cause a measurable power across the terminals of that impedance. A similarly accepted explanation on 1/*f*-noise is apparently not existing. It seems however to be evident from experiments (e.g. [1], [2], [3]) that 1/*f*-noise must be a phenomenon that is linked somehow to fluctuations of quantum states [4]. Moreover, it must be linked with the dissipation of power, since it is observed when a constant current is flowing thereby dissipating power.

It appears therefore to be sensible that for both 1/*f*-noise and thermal noise should exist a common explanation that is based on a theoretical description which considers fluctuations of quantum states.

Such a theory is available at least for cases where linear systems are to be considered: these are the (quantum) statistical mechanics of linear response. They are discussed intensively in text and reference books on statistics and quantum theory, e.g. [5] and [6].

Indeed, the only *quantum* theoretical model that explains thermal noise was found by an application of this theory. Early explanations go back to Callen and Welton [7]. They found for the first time an expression of the power spectrum of thermal noise that was completely derived from the quantum theoretical fluctuation-dissipation theorem (F-D theorem). Kubo [6] did not only reformulate the F-D theorem rigorously but also elaborated that the theorem was not only true for the description of “the dissipative part of the response to fluctuations but also to the dispersive (nondissipative) part”.

The question then arises why the F-D theorem does not yield useful results for 1/f-noise. Could it be that the applications of this theory were up to now subject to boundary conditions that are not met in the case of observation of 1/f-noise? Or could it be that the theorem was not correctly applied for the dispersive part?

It will be shown in this paper that at least the first question must be positively answered. However, if that is true, then it is very probable that the rigorous application of the F-D theorem or of the theory of linear response could also explain 1/f-noise. If that applies, then the way is open towards a unified theory for both thermal noise and 1/f-noise. The development of such a theory would be a new and yet unsolved problem on noise.

CRITQUE OF THE F-D THEOREM

There are two groups of models on thermal noise, the one group being variations of Nyquist’s model [8] and the other one being variations of Callen’s and Welton’s model [7].

Nyquist’s model was the attempt of explaining experiments that were performed by Johnson [9]. His experimental results could be well described by an assumed noise-power spectral density

$$N_{Johnson}\left(\omega/2\pi\right)=k\,T \tag{1}$$

where ω is the angular frequency, k is Boltzmann’s constant and T is the temperature in K. Nyquist’s first explanation [8] is essentially an application of the *classical* fluctuation-dissipation theorem (as it is described in chapter 2 of [6]). It predicted exactly Johnson’s noise power density. However, since this model would request infinite over-all-power, Nyquist adapted his model without giving any proof following Planck’s law of radiation:

$$N_{Nyquist}\left(\frac{\omega}{2\pi}\right)=\frac{\hbar\omega}{e^{\hbar\omega/kT}-1} \tag{2}$$

where $\hbar$ is Planck’s constant divided by 2π. Nyquist’s formula was published in 1928. Although it has no satisfying physical explanation from today’s point of view, it is widely accepted until now.

In 1951, Callen and Welton used the (quantum-theoretical) F-D theorem to analyze fluctuations of quantum states in an “impedance”. This theorem is elaborated in a more modern form in chapter 4 of [6]. It leads to a different expression for the power spectral density of thermal noise:

$$N_{CW}\left(\frac{\omega}{2\pi}\right)=\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}+\frac{\hbar\omega}{2} \quad . \tag{3}$$

The difference of this result to Nyquist's model is the additional term $\hbar\omega/2$ and that is sometimes referred to as the energy of "zero-point fluctuations". This dominates for high frequencies and leads to another ultraviolet catastrophe. This is why that model was disputed passionately in the past and why many scientists simply ignore this additional term though they agree that fluctuations of quantum states be the cause of thermal noise. Summaries of that discussion could be found in [10] and in [11]. It is not going to be repeated here.

The above discussion reveals a dilemma: Nyquist's model predicts an acceptable behavior for high frequencies but has no rigorous theoretical basis, while the F-D theorem opens the way towards a microscopic analysis, but predicts an unacceptable result for high frequencies.

An experienced way to overcome such a dilemma is to perform measurements. A closer look to more modern measurements shows that most of them were done with equipment that was calibrated under the assumption that Nyquist's formula was correct! Even if it is *assumed that Nyquist's formula is correct for sufficiently low frequencies*, then there do not exist many observations for high frequencies or for low temperatures that could help in deciding which of the models would match reality better. Such experiments were performed by Koch, van Harlingen and Clarke [12], Movshovich, Yurke, Kaminsky, Smith, Silver, Simon, and Schneider [13] and Schoelkopf, Burke, Kozhevnikov, Prober and Rooks [14]. Their results seem to corroborate that this last sum term must be interpreted as zero-point fluctuations of energy and to strongly support the opinion that in a frequency range between 19 GHz and 500 GHz and for temperatures from 30 mK to 4.2 K these particular setups could be well described by Callen's and Welton's model while the *Nyquist model fails* to describe it.

Experiments that analyze the absorption of photons in the optical range seem to yield another result. Gardiner [15] for example interprets the additional term $\hbar\omega/2$ as zero-point spectrum of black body radiation. He writes: "… we cannot detect the zero point spectrum when we measure the spectrum by means involving the absorption of photons from a radiation field." That means*: the F-D theorem fails* to describe thermal noise when the experimental setup includes a radiation field.

There is another fact at the opposite end of the spectrum that is comparably obscure: If it is claimed that the F-D theorem completely describes dissipation of energy in an impedance, why then does it not comprise a term that includes a constant current to describe the power dissipated on the average? The consequence is that it cannot possibly describe correctly noise at the low end of the spectrum. It should then not be a surprise that it fails to describe $1/f$-noise.

It must therefore be allowed to question where there could be weak points or inapplicable assumptions in the application of the F-D theorem.

THE F-D THEOREM IS SUBJECT TO LIMITING CONDITIONS

One possible weak point in the derivation of the F-D theorem could be hidden in the basic assumptions that where met at the very beginning of the derivation of the theorem. Callen and Welton [7] as well as Kubo [6] assumed that the system under consideration was a particle system described by a Hamilton operator

$$H = H_{\text{int}} + H_{\text{ext}} \tag{4}$$

where H_{int} is the operator when no external “forces” are acting on the system and

$$H_{\text{ext}} = \sum_i V_i(t)\, Q_i\left(\ldots, \vec{r}_\alpha, \ldots, \vec{p}_\alpha, \ldots\right) \tag{5}$$

is an operator describing small external perturbation forces. In this equation is $\left(\ldots, \vec{r}_\alpha, \ldots, \vec{p}_\alpha, \ldots\right)$ the set of generalized coordinates and canonically conjugate momenta of the particles. The particles are numbered by an index α.

However, consulting a modern textbook on quantum electrodynamics, e.g. [16], reveals that the operator for the external influences should be described by a much more complicated expression:

$$\begin{aligned} H_{\text{ext}} = {} & \sum_\alpha \frac{1}{m_\alpha}\left[-q_\alpha\left(\vec{p}_\alpha - q_\alpha \vec{A}_{\text{int}}(\vec{r}_\alpha, t)\right)\cdot \vec{A}_{\text{ext}}(\vec{r}_\alpha, t) + q_\alpha^2\, \vec{A}_{\text{ext}}^2(\vec{r}_\alpha, t)\right] + \sum_\alpha q_\alpha \Phi_{\text{ext}}(\vec{r}_\alpha, t) \\ & + \frac{\varepsilon_0}{2}\int d^3r \left[\left(\frac{\partial}{\partial t}\vec{A}_{\text{ext}}(\vec{r}, t)\right)^2 + c^2\left(\nabla\times\vec{A}_{\text{ext}}(\vec{r}, t)\right)^2\right] - \sum_\alpha \frac{g_\alpha q_\alpha}{2 m_\alpha}\vec{S}_\alpha \cdot \left\{\nabla\times\vec{A}_{\text{ext}}\right\}\Big|_{\vec{r}_\alpha} . \end{aligned} \tag{6}$$

In this equation mean

$$\begin{aligned} \vec{A}(\vec{r}, t) &= \vec{A}_{\text{ext}}(\vec{r}, t) + \vec{A}_{\text{int}}(\vec{r}, t), \\ \Phi(\vec{r}, t) &= \Phi_{\text{ext}}(\vec{r}, t) + \Phi_{\text{int}}(\vec{r}, t) \end{aligned} \tag{7}$$

the operators of a vector potential and a scalar potential that reduce to the operators with index “int” if no external fields exist. q_α is the charge of the particle with index α, m_α is its mass, g_α is the Landé-factor for the particle and $\vec{S}_\alpha$ the spin-operator associated to the particle with number α. ε_0 is the permittivity and c the velocity of light in vacuum. [1]

If it is assumed that the operator of the external vector potential is time-variant, then it follows that the Hamiltonian following equation (6) could possibly not be identified with the Hamiltonian following equations (4) and (5) that is used until now for the derivation of noise and fluctuations in an impedance: that Hamiltonian is too much simplifying. However, if there

[1] These are the operators commonly used in Maxwell’s theory or in quantum electrodynamics. In the above used formulation, Coulomb gauge is used as well for the internal potential functions as for the external potential functions.

are some simplifications possible, then the Hamiltonians will become very similar. This applies in the case of low-energy photons and sufficiently low radiation intensities.

In the appendix of [17] it is shown, that the interaction between spins and magnetic induction could be neglected against the first-order particle-field-interactions for low-energy photons, where the momentum of the photon is small compared with the momentum of the particles. This applies for example for the interaction between bound electrons and microwave photons. The influence of the spin-operator to the Hamiltonian will therefore no longer be considered.

In the same way, the second-order particle-field-interactions will be neglected against the first-order particle-field-interactions, since it will be supposed that radiation intensities are sufficiently low. The same argumentation assesses photon-photon interactions as negligible.

With these restrictions, the Hamiltonian of the system under consideration could be expressed as

$$H = H_0 + \sum_a q_a \left[\Phi_{\text{ext}}\left(\vec{r}_a, t\right) - \frac{1}{m_a} \vec{p}_a \cdot \vec{A}_{\text{ext}}\left(\vec{r}_a, t\right) \right] \quad . \tag{8}$$

This Hamiltonian is formally similar to that used by Kubo [6]. However, in Kubo's famous explanations on conductivity, a simpler Hamiltonian is demanded that could be found by neglecting the terms that depend on the vector potential (formula (4.5.8) in [6]). Callen and Welton [7] use a still more simple operator.

The terms depending on the vector potential, however, are directly correlated to the operator of current density. It could therefore be speculated that this is one possible reason why the model does not include statements on $1/f$-noise or on shot noise.

As a conclusion of this section, it follows that the F-D theorem is only applicable in the case of low-energy photons, sufficiently low radiation intensities and negligible current densities. That means: it is to be expected that this model will yield good results, if the average current density is 0 and if no radiation effects are to be taken into account. This could explain why the experiments in [12] and [13] are described well by the existing model, while optical experiments follow another rule. It could furthermore explain why at frequency 0 the results do not apply for a non-vanishing average current.

The same argumentation leads to the speculation that application of the dispersive part of the F-D theorem will not yield correct results in the existing model.

CONCLUSIONS AND OPEN QUESTIONS

It was shown in the above sections that models known from literature are likely to be inapplicable for the prediction of noise as well at high frequencies as at low frequencies. The reason for this is that for the derivation of the theorems used in those models, a too simplified Hamiltonian was used.

It is expected that as well fluctuations in power waves (thermal noise) as fluctuations in particle current (1/*f*-noise) could be predicted by proper application of an appropriate Hamilton operator in a model that uses the methods of linear response in quantum electrodynamics. Thus, a unified quantum theory of thermal noise and 1/*f*-noise should be possible. A consistent theory is however not yet available. Its development remains as a yet unsolved problem. There is a good chance that this problem could be solved in the near future. As a major result, it should be possible to answer the question whether thermal noise and 1/*f*-noise have the same physical causes, namely fluctuations of quantum states.

REFERENCES

1. Hooge, F.N., Kleinpenning, T.G.M., and Vandamme, L.K.J., *Rep. Prog. Phys.* **44,** 479-532 (1981)
2. van der Ziel, A., *Proceedings of the IEEE* **76,** 233-258 (1988)
3. Dutta, P., and Horn, P.M., *Rev. Mod. Phys.* **53,** 497-516 (1981)
4. Handel, P.H., *Phys. Rev. A* **22,** 745-757 (1980)
5. Landau, L.D., and Lifshitz, E.M., *Statistical Physics* , London: Pergamon Press, 1963.
6. Kubo, R., Toda, M., and Hashitsume, N., *Statistical Physics II - Nonequilibrium Statistical Mechanics* , Berlin, Heidelberg, New York, Tokyo: Springer, 1991.
7. Callen, H.W., and Welton, T.A., *Physical Review* **83,** 34-40 (1951)
8. Nyquist, H., *Physical Review* **32,** 110-113 (1928)
9. Johnson, J.B., *Physical Review* **32,** 97-109 (1928)
10. Abbott, D., Davis, B.R., Phillips, N.J., and Eshraghian, K., *Quantum Vacuum Fluctuations, Zero Point Energy and the Question of Observable Noise*. In: Doering, C.R., Kiss, L.B. and Shlesinger, M.F., *UPoN'96* , Singapore, New Jersey, London, Hong Kong: World Scientific, 1997, pp. 131-138.
11. Reggiani et al., *Quantum Noise in Transport Resistive Systems - Is It Detectable ?* In: Doering, C.R., Kiss, L.B., and Shlesinger, M.F., *UPoN'96* , Singapore, New Jersey, London, Hong Kong: World Scientific, 1997, pp. 139-143.
12. Koch, R.H., van Harlingen, D.J., and Clarke, J., *Phys. Rev. B* **26,** 74-87 (1982)
13. Movshovich et al., *Phys. Rev. Letters* **65,** 1419-1422 (1990)
14. Schoelkopf, R.J., Burke, P.J., Kozhevnikov, A.A., Prober, D.E., and Rooks, M.J., *Phys. Rev. Letters* **78,** 3370-3373 (1997)
15. Gardiner, C.W., *Quantum Noise* , Berlin, Heidelberg, New York: Springer, 1991.
16. Cohen-Tannoudji, C., Dupont-Roc, J., and Grynberg, G., *Photons and Atoms* , New York: Wiley, 1989.
17. Cohen-Tannoudji, C., Dupont-Roc, J., and Grynberg, G., *Atom-Photon Interactions* , New York: Wiley, 1992.