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Derivation of a Quantum-Electrodynamical Hamiltonian for Describing Noise Processes

Michael H.W. Hoffmann

University of Ulm, Department of Microwave Techniques, Albert-Einstein-Allee 41, Germany
E-mail: Michael.Hoffmann@e-technik.uni-ulm.de

Abstract. Classical models on electronic noise do not yield satisfying results. This applies even for thermal noise, since classical models predict a constant noise power density for all frequencies which requires the availability of infinitely large energy. It is thus necessary to use quantum-theoretical models. These are always based on the formulation of a Hamiltonian operator that describes the system under consideration in a sufficiently accurate approach. Hamiltonians that are used in the literature to analyze electronic noise are either very special (see for example the Grau-Kleen operator) or they are very general (see for instance the Kubo operator). Therefore, a Hamiltonian is derived that matches the needs for an analysis of electronic noise in a solid-state impedance. The derivation starts with the derivation of a Hamiltonian in a classical multi-particle system by applying the Newton-Lorentz equations and Maxwell's equations. The independently varying quantities are identified by use of the Lagrange formalism. It is seen that in addition to the space coordinates and momenta of the (charged) particles, field-components will appear that can be derived from a magnetic vector potential. In the derived classical Hamiltonian, infinitely large self-energy does not appear. In a next step, a quantum-electrodynamical Hamiltonian is found by introducing commutation relations. Quantities derived from the vector potential turn into the creation and annihilation operator. The newly found Hamiltonian is compared to the Grau-Kleen operator and the Kubo operator. It shows that it includes both of them as special cases. The new formulas gives rise to the hope that 1/*f*-noise could be explained by application of the novel Hamiltonian.

1 Introduction

Noise-generating systems are subject to intensive physical research work since more than eighty years [1]. For a better understanding of the physics of such a system, in particular for quantum 1/*f*-noise, a quantum model must be used. An essential part of the latter is a Hamiltonian operator.

This insight is not new. Many authors have used different approaches to Hamiltonians for systems with noise, e.g. [2–5]. It seems, however, that a derivation is still missing that describes exactly under which conditions the model holds. Therefore, an attempt is made to construct a suitable Hamiltonian rigorously (within a given model) from basic physical laws.

The procedure is straightforward: First a classical Lagrangian is found for a system consisting of charged mass particles in an electromagnetic field. The system is then subdivided into subsystems, one of them being the "impedance" that produces noise, the other one being its environment. The derivation of these Lagrangians is necessary to find independent canonical coordinates and momenta. Then classical Hamiltonians will be derived. Finally, quantum-electrodynamical Hamiltonian operators are found by canonical quantization. This procedure will not only show the restrictions under which the model is valid, but it will also unify models that are seen as competing models until now.

2 The classical Lagrangian

In this section, we will consider a classical system consisting of many mass particles with (possibly different) charges in an electromagnetic field.

2.1 Short recall of Lagrange's formulation

To begin with, a short recall of Lagrange's formulation will be given that is mainly based on Goldstein's introduction into classical mechanics [6].

Lagrange's equations are nothing else than an advantageous formulation of the dynamical equations for a given system. It is assumed that the dynamics of the system are described completely by a set of second-order differential equations, e.g. the equations of motion and wave equations, and possibly by some other equations that outline constraints. The latter reduce the degrees of freedom. Therefore, a complete set of *independent variables* is looked for that determines the dynamics of the system without violating the constraints. These independent variables will be named q_i. The dynamics of the system will thus be completely known, if first-order differential equations are solved for these variables *and* for their first derivatives with respect to time. Since the new variables are not necessarily coordinates in the common sense, they will be called generalized or canonical coordinates. Following the same argumentation, their derivatives will be called generalized or canonical velocities.

Differentially small pieces of work that are executed in the system can be formulated as dot products of a force-vector and a differentially small displacement vector. If these pieces of work are rewritten in terms of generalized coordinates, then they can be reformulated as dot products of a differentially small vector in terms of generalized coordinates and a new vector that will be called generalized force.

Under the condition that the constraints are holonomic and that generalized forces are obtained from scalar "potential-functions" (that might be velocity-dependent), it is possible to derive the complete dynamics from a scalar function L that is called the Lagrangian of the system. Lagrange's equations are then given as

$$d\left(\partial L/\partial \dot{q}_i\right)\Big/dt-\partial L/\partial q_i=0 \tag{1}$$

in case of countable many real-valued generalized coordinates. In case of complex-valued canonical variables, this set of equations is completed by the following set of equations:

$$d\left(\partial L \Big/ \partial \overset{\bullet}{q}{}_{\nu}^{*}\right)\Big/ dt - \partial L / \partial q_{\nu}^{*} = 0 \ . \tag{2}$$

The quantities

$$p_i := \partial L \Big/ \partial \overset{\bullet}{q}{}_{\nu}^{*} \tag{3}$$

will be called canonical or conjugate momenta.

2.2 Electrodynamics

If a system of N charged particles is to be described, then the force acting on the particle with index number n is

$$\vec{F}_n = \Delta q_n \left(\vec{E}_{(n)}\left(\vec{r}_n, t\right) + \overset{\bullet}{\vec{r}}_n \times \vec{B}\left(\vec{r}_n, t\right)\right) \tag{4}$$

where $\Delta q_n \vec{E}_{(n)}$ is the force that the electric field imposes on the n-th charged particle and $\Delta q_n \vec{r}_n \times \vec{B}$ is the Lorentz-force. Thus, it will not be possible to express this force as a function of a (possibly velocity-dependent) potential without further knowledge on the fields. These are described by Maxwell's equations:

$$\nabla \cdot \vec{B}\left(\vec{r}, t\right) = 0 \ , \tag{5}$$

$$\nabla \times \vec{E}\left(\vec{r}, t\right) = -\frac{\partial \vec{B}\left(\vec{r}, t\right)}{\partial t} \ , \tag{6}$$

$$\nabla \cdot \vec{E}\left(\vec{r}, t\right) = \frac{1}{\varepsilon_0} \rho\left(\vec{r}, t\right) \ , \tag{7}$$

$$\nabla \times \vec{B}\left(\vec{r}, t\right) = \frac{1}{c^2} \frac{\partial \vec{E}\left(\vec{r}, t\right)}{\partial t} + \frac{1}{c^2 \varepsilon_0} \vec{j}_C\left(\vec{r}, t\right) \ . \tag{8}$$

The following method to solve these equations is strongly influenced by expositions found in [7] and [8].

To get rid of the nabla-operator, it will be assumed that the equations will be solved in ordinary space only within a cube of finite (but very large) length L_{cube}. Within this cube, any power-limited function $U\left(\vec{r}, t\right)$ can be approached with arbitrary accuracy by a spatial Fourier series:

$$U\left(\vec{r}, t\right) = \sum_{l_1, l_2, l_3 = -\infty}^{\infty} \breve{U}\left(\vec{k}, t\right) e^{j\vec{k} \cdot \vec{r}} \quad ; \quad \vec{k} = \frac{2\pi}{L_{cube}}\left(l_1 \vec{e}_1 + l_2 \vec{e}_2 + l_3 \vec{e}_3\right) \tag{9}$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ form an orthonormal set of unit vectors in ordinary space and with

$$\breve{U}\left(\vec{k},t\right)=\frac{1}{L^3}\int\limits_{(cube)} d^3r\, U\left(\vec{r},t\right)e^{-j\vec{k}\cdot\vec{r}} \quad . \tag{10}$$

Vectors $\vec{k}$ span a space that is known as reciprocal space. If L_{cube} tends towards infinity, then the series expansion tends towards the Fourier spatial transform.

It will be assumed that the magnetic induction and the electric field can be expanded into spatial Fourier series. Because of the orthogonality of the exponential functions it follows:

$$j\vec{k}\cdot\breve{\vec{B}}\left(\vec{k},t\right)=0 \ , \tag{11}$$

$$j\vec{k}\times\breve{\vec{E}}\left(\vec{k},t\right)=-\dot{\breve{\vec{B}}}\left(\vec{k},t\right) \ , \tag{12}$$

$$j\vec{k}\cdot\breve{\vec{E}}\left(\vec{k},t\right)=\frac{1}{\varepsilon_0}\breve{\rho}\left(\vec{k},t\right) \ , \tag{13}$$

$$j\vec{k}\times\breve{\vec{B}}\left(\vec{k},t\right)=\frac{1}{c^2}\dot{\breve{\vec{E}}}\left(\vec{k},t\right)+\frac{1}{c^2\varepsilon_0}\breve{\vec{j}}_C\left(\vec{k},t\right) \ , \tag{14}$$

In the following, the argument will be omitted, if possible.

Equation (11) shows that $\breve{\vec{B}}$ must be orthogonal to $\vec{k}$. Therefore, if we exclude the irrelevant case $\breve{\vec{B}}(\vec{0})\neq\vec{0}$, it must be possible to obtain $\breve{\vec{B}}$ as

$$\breve{\vec{B}}=j\vec{k}\times\breve{\vec{A}} \quad \Leftrightarrow \quad \vec{B}=\nabla\times\vec{A} \ , \tag{15}$$

$\breve{\vec{A}}$ is known as the (Fourier series coefficient of) the vector potential. It is not yet completely determined by equation (15), since it could be completed by any component that is parallel to the vector $\vec{k}$. We have thus the freedom to choose

$$j\vec{k}\cdot\breve{\vec{A}}=0 \ , \tag{16}$$

without affecting the result. This choice is known as *Coulomb gauge* or *radiation gauge.*

Insertion of equation (15) into equation (12) yields

$$j\vec{k}\times\left\{\breve{\vec{E}}+\dot{\breve{\vec{A}}}\right\}=\vec{0} \ , \tag{17}$$

which means that the vector in the curly brackets must be parallel to the vector $\vec{k}$ for any argument. If we exclude the irrelevant case $\breve{\vec{E}}\left(\vec{0}\right)\neq\vec{0}$, it must be possible to construct the vector in curly brackets as:

$$\breve{\vec{E}}+\dot{\breve{\vec{A}}}=-j\vec{k}\,\breve{\phi} \ . \tag{18}$$

The factor $-j$ is due to conventions in literature. $\breve{\phi}$ is known as the (Fourier coefficient of the) scalar potential. The electric field may thus be substituted by the following combination

$$\breve{\vec{E}} = -\dot{\breve{\vec{A}}} - j\vec{k}\,\breve{\phi} \quad \Leftrightarrow \quad \vec{E} = -\partial\vec{A}/\partial t - \nabla\phi \ , \tag{19}$$

Thus, by introduction of the vector potential and the scalar potential, equations (11) and (12) will be met automatically.

Inserting these potentials and the Coulomb gauge into equation (13) yields (in the limiting case where the Fourier series tends to the Fourier transform)

$$\breve{\phi} = \frac{1}{\varepsilon_0 k^2}\breve{\rho} \quad \Leftrightarrow \quad \phi = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho\left(\vec{r}',t\right)}{\left|\vec{r}'-\vec{r}\right|} \ . \tag{20}$$

This equation indicates clearly that with known charge density, the scalar potential is determined completely. Thus, $\breve{\phi}$ cannot be an independent variable. Within the frame of this derivation, equations (11) - (13) may thus be interpreted as equations describing *constraints*.

Since the Coulomb gauge was used, the electric field will be separated by equation (19) into two orthogonal terms:

$$\breve{\vec{E}} = \breve{\vec{E}}_\perp + \breve{\vec{E}}_\parallel \quad \text{with} \quad \breve{\vec{E}}_\perp = -\dot{\breve{\vec{A}}} \quad \text{and} \quad \breve{\vec{E}}_\parallel = -j\vec{k}\,\breve{\phi} \quad , \tag{21}$$

The notation follows that given by Messiah [9]. For an easier formulation, we introduce an orthonormal system of unit vectors for each particular vector $\vec{k}$:

$$\vec{e}_{-1} := \frac{\vec{k}}{k} \ ; \ k = \left|\vec{k}\right| \ ; \ \vec{k} = k_{-1}\,\vec{e}_{-1} = k\,\vec{e}_{-1} \ ; \ \vec{e}_i \cdot \vec{e}_l = \delta_{i,l} \ ; \ i,l = -1,-2,-3 \ , \tag{22}$$

Indices were chosen to be negative in order not to confuse these unit vectors with the unit vectors given earlier. The proper directions of the vectors $\vec{e}_{-2}$ and $\vec{e}_{-3}$ are not yet fixed. With these unit vectors, it will be possible to decompose any vector $\vec{V}$ into a sum of a vector $\vec{V}_\parallel$ that is parallel to $\vec{k}$ (the “longitudinal” component) and another one, $\vec{V}_\perp$, that is orthogonal to $\vec{k}$ (the “transversal” component):

$$\vec{V}_\parallel = \left(\vec{e}_{-1}\cdot\vec{V}\right)\vec{e}_{-1} \ ; \ \vec{V}_\perp = \vec{V} - \left(\vec{e}_{-1}\cdot\vec{V}\right)\vec{e}_{-1}, \tag{23}$$

Therefore:

$$\breve{\vec{A}} = \breve{\vec{A}}_\perp \ ; \ \breve{\vec{B}} = \breve{\vec{B}}_\perp \ , \tag{24}$$

It remains to analyze the last of Maxwell’s equations. Inserting the above results into equation (14) yields

$$c^2 j\vec{k}\times\left(j\vec{k}\times\breve{\vec{A}}_\perp\right)=-\ddot{\breve{\vec{A}}}_\perp-j\vec{k}\frac{1}{\varepsilon_0 k^2}\dot{\breve{\rho}}+\frac{\breve{\vec{j}}_C}{\varepsilon_0}\ , \tag{25}$$

Application of vector theorems results in

$$\ddot{\breve{\vec{A}}}_\perp+c^2k^2\breve{\vec{A}}_\perp=-j\vec{k}\frac{1}{\varepsilon_0 k^2}\dot{\breve{\rho}}+\frac{1}{\varepsilon_0}\breve{\vec{j}}_C\ , \tag{26}$$

Dot product with $j\vec{k}$ leads to

$$\frac{\partial}{\partial t}\breve{\rho}=-j\vec{k}\cdot\breve{\vec{j}}_C \quad\Leftrightarrow\quad \frac{\partial}{\partial t}\rho=-\nabla\cdot\vec{j}_C\ , \tag{27}$$

Equation (27) is nothing else than the continuity equation: Sources of a current of charged carriers are time-varying charge densities exclusively.

The transversal part of equation (26) reads as

$$\ddot{\breve{\vec{A}}}_\perp+c^2k^2\breve{\vec{A}}_\perp=\frac{1}{\varepsilon_0}\breve{\vec{j}}_{C\perp}\ , \tag{28}$$

or expressed in components as

$$\varepsilon_0\ddot{\breve{A}}_i+\varepsilon_0 c^2k^2\breve{A}_i=\breve{j}_{C,i}\quad;\quad i=-2,-3\ . \tag{29}$$

These are (for each individual vector $\vec{k}$) two second-order differential equations that determine the dynamics of the Fourier coefficients of the vector potential. They are thus essential for the determination of a Lagrangian.

To solve the last equations, we reformulate them as follows

$$\frac{d}{dt}\begin{pmatrix}\breve{A}_i\\ \varepsilon_0\dot{\breve{A}}_i\end{pmatrix}=\begin{pmatrix}0 & 1/\varepsilon_0\\ -\varepsilon_0c^2k^2 & 0\end{pmatrix}\begin{pmatrix}\breve{A}_i\\ \varepsilon_0\dot{\breve{A}}_i\end{pmatrix}+\begin{pmatrix}0\\ \breve{j}_{C,i}\end{pmatrix}=:\vec{\vec{M}}\begin{pmatrix}\breve{A}_i\\ \varepsilon_0\dot{\breve{A}}_i\end{pmatrix}+\begin{pmatrix}0\\ \breve{j}_{C,i}\end{pmatrix}\ ;\ i=-2,-3\ . \tag{30}$$

These are two first order differential equations for two components of the vector $(\breve{A}_i,\varepsilon_0\dot{\breve{A}}_i)^T$. We will map this vector by a linear time-invariant transformation matrix into a vector $\left(\breve{X}_i,\breve{Y}_i\right)^T$ in order to obtain a diagonalized matrix:

$$\begin{pmatrix}\breve{A}_i\\ \dot{\breve{A}}_i\end{pmatrix}=:\vec{\vec{T}}\begin{pmatrix}\breve{X}_i\\ \breve{Y}_i\end{pmatrix}\quad;\quad i=-2,-3\ . \tag{31}$$

Then it follows from equation (38)

$$\frac{d}{dt}\begin{pmatrix}\breve{X}_i\\ \breve{Y}_i\end{pmatrix}=\vec{\vec{T}}^{-1}\vec{\vec{M}}\ \vec{\vec{T}}\begin{pmatrix}\breve{X}_i\\ \breve{Y}_i\end{pmatrix}+\vec{\vec{T}}^{-1}\begin{pmatrix}0\\ \breve{j}_{C,i}\end{pmatrix}\quad;\quad i=-2,-3\ . \tag{32}$$

The case $k=0$ is irrelevant in our case, we choose therefore the transformation matrix in a way that

$$\vec{T}^{-1}\,\vec{M}\,\vec{T}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\;. \tag{33}$$

This procedure decouples the equations in a way that they can be solved immediately. Finding such a transformation matrix is a typical eigenvector-problem, since the column vectors of $\vec{T}$ are the eigenvectors of $\vec{M}$ and these are

$$\vec{v}_1=\left(-j/\varepsilon_0 ck\,,1\right)^T \quad \text{with eigenvalue} \quad \lambda_1=+jkc\;, \tag{34}$$

$$\vec{v}_2=\left(+j/\varepsilon_0 ck\,,1\right)^T \quad \text{with eigenvalue} \quad \lambda_2=-jkc\;. \tag{35}$$

Using this procedure, we obtain the eigenfunctions

$$\breve{X}_i=+j\frac{\varepsilon_0\, c\, k}{2}\,\breve{A}_i+\frac{1}{2}\,\varepsilon_0\,\dot{\breve{A}}_i \quad ; \quad i=-2,-3 \tag{36}$$

$$\breve{Y}_i=-j\frac{\varepsilon_0\, c\, k}{2}\,\breve{A}_i+\frac{1}{2}\,\varepsilon_0\,\dot{\breve{A}}_i \quad ; \quad i=-2,-3 \tag{37}$$

and from that the general solution of the second-order differential equations:

$$\breve{\vec{A}}_\perp=\frac{j}{2\varepsilon_0 ck}\int_0^t \breve{\vec{j}}_C\left(\vec{k},\tau\right)\left\{e^{-jkc(t-\tau)}-e^{jkc(t-\tau)}\right\}d\tau+\vec{K}_{1\perp}\left(\vec{k}\right)e^{+jkct}+\vec{K}_{2\perp}\left(\vec{k}\right)e^{-jkct}\;. \tag{38}$$

In the homogeneous case, these solutions represent waves that propagate parallel or antiparallel to the vector $\vec{k}$. This vector is therefore called propagation vector and equation (28) is a transformed wave equation. Note that it is not necessary to assume that these waves are sinusoidal waves. Therefore, *an assignment of a frequency would be senseless at this point*.

Since the homogeneous equation has non-vanishing solutions, it must be concluded from equation (38) that even in the case where there are no moving charges present in the space, the vector potential and thus the electromagnetic field could be different from zero. That means: even in case of a free space without charges, there could exist field waves. In the quantized case, these solutions are photons.

In the general case, the wave equations are inhomogeneous second order differential equations with constant coefficients that determine the dynamical behavior of the minus-second and the minus-third component of the vector potential in reciprocal space. It is therefore suspected that those components of the vector potential could be dynamical variables in the sense of the Lagrangian formalism.

There is, however, a problem. Since $\breve{\vec{A}}$ is a Fourier coefficient of a real-valued function, we have

$$\breve{\vec{A}}\left(-\vec{k},t\right)=\breve{\vec{A}}^*\left(\vec{k},t\right)\;. \tag{39}$$

That means: Only half of all possible variables $\breve{\vec{A}}\left(\vec{k},t\right)$ could be independent variables. Therefore, equation (29) is split into two equations: one for positive values of k_{+1} and another one for negative values of k_{+1}. Then the above equation is applied to result in

$$\varepsilon_0 \ddot{\breve{A}}_i + \varepsilon_0 c^2 k^2 \breve{A}_i = \breve{j}_{C,i} \quad ; \quad k_1 > 0 \quad ; \quad i = -2,-3 \ . \tag{40}$$

$$\varepsilon_0 \ddot{\breve{A}}_i^* + \varepsilon_0 c^2 k^2 \breve{A}_i^* = \breve{j}_{C,i}^* \quad ; \quad k_1 > 0 \quad ; \quad i = -2,-3 \ . \tag{41}$$

These equations bear all information that was available in equation (29).

Next, a function $\breve{L}_{el}\left(\vec{k}\right)$ is defined as

$$\breve{L}_{el} = \varepsilon_0 \dot{\breve{\vec{A}}} \cdot \dot{\breve{\vec{A}}}^* - \varepsilon_0 c^2 \left(j\vec{k} \times \breve{\vec{A}}\right) \cdot \left(j\vec{k} \times \breve{\vec{A}}\right)^* + \breve{\vec{A}} \cdot \breve{\vec{j}}_C^* + \breve{\vec{A}}^* \cdot \breve{\vec{j}}_C \ . \tag{42}$$

Using this function, it is not too difficult to show that the wave equations (40) and their complex conjugates can be written as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\breve{A}}_i}\breve{L}_{el}\right) = \frac{\partial}{\partial \breve{A}_i}\breve{L}_{el} \ ; \ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\breve{A}}_i^*}\breve{L}_{el}\right) = \frac{\partial}{\partial \breve{A}_i^*}\breve{L}_{el} \ ; \ k_1 > 0 \ ; \ i = -2,-3 \ , \tag{43}$$

which is a set of Lagrangian equations with the Lagrangian $\breve{L}_{el}$ and infinitely many independent canonical variables $\breve{A}_i\left(\vec{k},t\right)$ and $\breve{A}_i^*\left(\vec{k},t\right)$, $k_1>0$. Summation over all possible values of $\vec{k} = 2\pi\left(l_1\vec{e}_1 + l_2\vec{e}_2 + l_3\vec{e}_3\right)/L$ with positive k_1 leads to a new Lagrangian

$$\begin{aligned} L_{el} &= \sum_{\substack{l_1>0,\\ l_2,l_3=-\infty}}^{\infty} \breve{L}_{el}\left(2\pi\left(l_1\vec{e}_1 + l_2\vec{e}_2 + l_3\vec{e}_3\right)/L, t\right) \\ &= \sum_{\substack{l_1>0,\\ l_2,l_3=-\infty}}^{\infty} \left\{\varepsilon_0 \dot{\breve{\vec{A}}} \cdot \dot{\breve{\vec{A}}}^* - \varepsilon_0 c^2 \left(j\vec{k} \times \breve{\vec{A}}\right) \cdot \left(j\vec{k} \times \breve{\vec{A}}\right)^* + \breve{\vec{A}} \cdot \breve{\vec{j}}_C^* + \breve{\vec{A}}^* \cdot \breve{\vec{j}}_C\right\} \\ &= \frac{1}{2} \sum_{l_1,l_2,l_3=-\infty}^{\infty} \left\{\varepsilon_0 \dot{\breve{\vec{A}}} \cdot \dot{\breve{\vec{A}}}^* - \varepsilon_0 c^2 \left(j\vec{k} \times \breve{\vec{A}}\right) \cdot \left(j\vec{k} \times \breve{\vec{A}}\right)^* + \breve{\vec{A}} \cdot \breve{\vec{j}}_C^* + \breve{\vec{A}}^* \cdot \breve{\vec{j}}_C\right\} \end{aligned} \quad , \tag{44}$$

where the last sum is performed over the complete reciprocal space, i.e. without any restrictions concerning the values of l_1. In general, L_{el} is not yet a complete (classical) Lagrangian for the system of charged particles, since the dynamics of mass particles are not yet included.

Applications of Parseval's identity (for Fourier series) to the sum defining L_{el} yields

$$L_{el}=\int\frac{\varepsilon_0}{2}\left[\dot{\vec{A}}\cdot\dot{\vec{A}}-c^2\left(\nabla\times\vec{A}\right)\cdot\left(\nabla\times\vec{A}\right)\right]d^3r+\int\vec{A}\cdot\vec{j}_C\,d^3r\ . \tag{45}$$

Note that in the last equation all functions are defined in ordinary space.

2.3 Mechanical dynamics of mass particles

Up to now, only the field dynamics have been analyzed. Now, the mechanical dynamics of mass particles are to be included. Newton's second law of motion describes the dynamics of mass particles. For the n-th particle, we have the simple relation

$$\Delta m_n\,\ddot{\vec{r}}_n=\vec{F}_n\,, \tag{46}$$

where $\vec{F}_n$ is the force acting on the particle with index n. To compute this force, it will be assumed that only point charges with mass Δm_n and electric charge Δq_n will be considered. Their positions be $\vec{r}_n\left(t\right)$. (The dependence of t will only be indicated in the following, if necessary). Then charge density and current density will be given as

$$\rho\left(\vec{r},t\right)=\sum_{l=1}^{N}\Delta q_n\ \delta\left(\vec{r}-\vec{r}_l\right)\ , \tag{47}$$

$$\vec{j}_C\left(\vec{r},t\right)=v\left(\vec{r},t\right)\rho\left(\vec{r},t\right)=\sum_{l=1}^{N}\dot{\vec{r}}_l\,\Delta q_l\ \delta\left(\vec{r}-\vec{r}_l\right) \tag{48}$$

From equation (20) it follows then

$$\phi\left(\vec{r},t\right)=\frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho\left(\vec{r}',t\right)}{\left|\vec{r}'-\vec{r}\right|}=\frac{1}{4\pi\varepsilon_0}\sum_{l=1}^{N}\frac{\Delta q_l}{\left|\vec{r}_l-\vec{r}\right|} \tag{49}$$

That means: the scalar potential is divergent at the position of the point charge. Therefore, we must be careful in making use of the scalar potential at points of singularity. And this is the point were we depart from the path that was taken by Cohen-Tannoudji, Dupont-Roc and Grynberg in [7–9] or others.

The longitudinal part of the electric field is

$$\vec{E}_{\parallel}=-\nabla\phi=\frac{1}{4\pi\varepsilon_0}\sum_{l=1}^{N}\frac{\Delta q_l}{\left|\vec{r}-\vec{r}_l\right|^2}\,\frac{\vec{r}-\vec{r}_l}{\left|\vec{r}-\vec{r}_l\right|} \tag{50}$$

and the complete electric field is

$$\vec{E}=\vec{E}_{\perp}+\vec{E}_{\parallel}=-\dot{\vec{A}}+\frac{1}{4\pi\varepsilon_0}\sum_{l=1}^{N}\frac{\Delta q_l}{\left|\vec{r}-\vec{r}_l\right|^2}\,\frac{\vec{r}-\vec{r}_l}{\left|\vec{r}-\vec{r}_l\right|}\quad . \tag{51}$$

However, if the force is considered that acts on the particle with index n, then we must respect the fact that a charged *point mass* cannot impose a force on itself! Thus, only the *punctuated* electric field

$$\vec{E}_{(n)}(\vec{r},t) = -\dot{\vec{A}}(\vec{r},t) + \frac{1}{4\pi\varepsilon_0}\sum_{\substack{l=1\\ l\neq n}}^{N}\frac{\Delta q_l}{|\vec{r}-\vec{r}_l|^2}\frac{\vec{r}-\vec{r}_l}{|\vec{r}-\vec{r}_l|} \tag{52}$$

is mediating a force on particle n. Neglecting all forces but the forces imposed by the electric and the magnetic field, we end up with a modified Newton-Lorentz equation:

$$\Delta m_n \ddot{\vec{r}}_n = \Delta q_n \left(\vec{E}_{(n)}(\vec{r}_n,t) + \dot{\vec{r}}_n \times \vec{B}(\vec{r}_n,t)\right). \tag{53}$$

Note that by this formulation, we have avoided any singularity in the equation of motion. Using the above-derived knowledge on fields, it follows

$$\Delta m_n \ddot{\vec{r}}_n = \Delta q_n \left\{-\frac{\partial\vec{A}(\vec{r}_n,t)}{\partial t} - \nabla\phi_{(n)}(\vec{r}_n,t) + \dot{\vec{r}}_n \times \left[\left(\nabla\times\vec{A}\right)(\vec{r}_n,t)\right]\right\} \tag{54}$$

with the *punctuated* scalar potential

$$\phi_{(n)}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{k=1\\ k\neq n}}^{N}\frac{\Delta q_k}{|\vec{r}-\vec{r}_k|} \quad . \tag{55}$$

From here on, the reformulation of the equations of motion for the charged particles as Lagrange equations is again exactly as described in [6], apart from the fact that instead of the complete scalar potential we have to use the punctuated scalar potential. We end up in a Lagrangian

$$L_{mech} = \int \vec{A}\cdot\vec{j}_C\, d^3r + \sum_{k=1}^{N}\Delta m_k \frac{\dot{\vec{r}}_k\cdot\dot{\vec{r}}_k}{2} - \frac{1}{8\pi\varepsilon_0}\sum_{\substack{k,l=1\\ k\neq l}}^{N}\frac{\Delta q_l \Delta q_k}{|\vec{r}_l-\vec{r}_k|} \tag{56}$$

the associated Lagrange equations of which are

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r}_{n,i}}L_{mech} = \frac{\partial}{\partial r_{n,i}}L_{mech} \quad . \tag{57}$$

Insertion of equation (56) then shows that these are indeed the equations of motion for the charged particles.

2.4 The complete Lagrangian

Comparison of the above Lagrangian and the Lagrangian L_{el} shows that there is one term in common. This is the integral containing the current density. Since the other sum terms in L_{el} are not dependent on the charge density or the current density, their partial derivatives with respect to either component of the position vectors of the particles or either component of their velocity vectors vanish. Therefore, the completed Lagrangian

$$
\begin{aligned}
L = & \int \left\{ \frac{\varepsilon_0}{2} \left[\dot{\vec{A}} \cdot \dot{\vec{A}} - c^2 \left(\nabla \times \vec{A}\right) \cdot \left(\nabla \times \vec{A}\right) \right] + \vec{A} \cdot \vec{j}_C \right\} d^3 r \\
& + \sum_{n=1}^{N} \Delta m_n \frac{\dot{\vec{r}}_n \cdot \dot{\vec{r}}_n}{2} - \frac{1}{8\pi\varepsilon_0} \sum_{\substack{m,n=1 \\ m \neq n}}^{N} \frac{\Delta q_n \Delta q_m}{\left|\vec{r}_n - \vec{r}_m\right|}
\end{aligned}
\tag{58}
$$

returns also the equations of motion of the charged particles by application of the Lagrange equations

$$
\frac{d}{dt} \frac{\partial}{\partial \dot{r}_{n,i}} L = \frac{\partial}{\partial r_{n,i}} L \quad . \tag{59}
$$

If Parseval’s theorem is applied to the Lagrangian following equation (58), then it follows

$$
\begin{aligned}
L = & \frac{1}{2} \sum_{l_1, l_2, l_3 = -\infty}^{\infty} \left\{ \varepsilon_0 \, \dot{\breve{\vec{A}}} \cdot \dot{\breve{\vec{A}}}^{*} - \varepsilon_0 c^2 \left(j\vec{k} \times \breve{\vec{A}} \right) \cdot \left(j\vec{k} \times \breve{\vec{A}} \right)^{*} + \breve{\vec{A}} \cdot \breve{\vec{j}}_C^{*} + \breve{\vec{A}}^{*} \cdot \breve{\vec{j}}_C \right\} \\
& + \sum_{n=1}^{N} \frac{\Delta m_n}{2} \dot{\vec{r}}_n \cdot \dot{\vec{r}}_n - \frac{1}{8\pi\varepsilon_0} \sum_{\substack{k,l=1 \\ k \neq l}}^{N} \frac{\Delta q_l \Delta q_k}{\left|\vec{r}_l - \vec{r}_k\right|}
\end{aligned}
\tag{60}
$$

This shows that the Lagrange equations

$$
\begin{aligned}
& \frac{d}{dt} \left(\frac{\partial}{\partial \dot{\breve{A}}_i \left(\vec{k}, t\right)} L \right) = \frac{\partial}{\partial \breve{A}_i \left(\vec{k}, t\right)} L \quad ; \\
& \frac{d}{dt} \left(\frac{\partial}{\partial \dot{\breve{A}}_i^{*} \left(\vec{k}, t\right)} L \right) = \frac{\partial}{\partial \breve{A}_i^{*} \left(\vec{k}, t\right)} L \quad ; \quad i = -2, -3
\end{aligned}
\quad ,
\tag{61}
$$

return the wave equations for the components of the vector potential. Thus, the Lagrangian following equation (58) or (60) is suitable to describe all necessary dynamic equations.

This Lagrangian differs from those given in literature in one *essential point*: the Coulomb energy term does not contain infinitely large contributions! There is no self-energy term present. This will make possible a mathematical rigorous renormalization of the zero-point energy.

Summarizing the restrictions that have been used to derive this result, we state that

1. No other forces than electromagnetic forces have been taken into consideration.
2. No relativistic effects are considered.
3. It was assumed that a point charge could not impose a force on itself.
4. Masses and charges were considered to be point masses and point charges. Thus, no rotational effects of charged particles can be derived (spins !).

Would we use a model where the charges are not concentrated in point masses, then the vectors $\vec{r}_n$ had to be interpreted as charge centers. An additional force would then have to be respected, that conserves the charge distribution of the particles. Besides others, the (classical) Lagrangian had furthermore to be completed by a term that stems from the rotational energy and another one that takes into account the interaction between rotating charge and magnetic field. If we neglect these additional terms, then we should not be surprised to miss any effects that are related to spins of particles.

3 The classical Lagrangian for a system that is controlled by external forces

We assume now that the set of all charged particles be separated into two disjoint sets. One of the sets could configure an "impedance" the noise behavior of which shall be analyzed, the other one could form the environment of that system. Thus the vector of current density could be split into the superposition of two current densities, charge density could be separated into two sum terms:

$$\vec{j}_C(\vec{r},t) = \vec{j}_C^{(I)}(\vec{r},t) + \vec{j}_C^{(II)}(\vec{r},t) \tag{62}$$

$$\rho(\vec{r},t) = \rho^{(I)}(\vec{r},t) + \rho^{(II)}(\vec{r},t) \tag{63}$$

$$\vec{j}_C^{(I)}(\vec{r},t) = \sum_{n=1}^{N_1} \dot{\vec{r}}_n \, \Delta q_n \, \delta(\vec{r}-\vec{r}_n) \quad ; \quad \vec{j}_C^{(II)}(\vec{r},t) = \sum_{n=N_1+1}^{N} \dot{\vec{r}}_n \, \Delta q_n \, \delta(\vec{r}-\vec{r}_n) \tag{64}$$

$$\rho^{(I)}(\vec{r},t) = \sum_{n=1}^{N_1} \Delta q_n \, \delta(\vec{r}-\vec{r}_n) \quad ; \quad \rho^{(II)}(\vec{r},t) = \sum_{n=N_1+1}^{N} \Delta q_n \, \delta(\vec{r}-\vec{r}_n) \tag{65}$$

where the superscript (I) identifies particles from set I and the superscript (II) identifies particles from set II.

Our main concern is the derivation of equations of motion and of the wave equations that govern the particles in the first of both sets. We will therefore try to find a Lagrangian from which these equations could be derived. Lagrange equations for the second set of equations will not be treated.

We start with the wave equations and make use of the above splitting of current densities while reconsidering the second-order differential equations that describe the dynamics of the vector potential:

$$\varepsilon_0 \frac{\partial^2 \breve{A}_i}{\partial t^2} + \varepsilon_0 c^2 k^2 \breve{A}_i = \breve{j}_{C,i}^{(I)} + \breve{j}_{C,i}^{(II)} \quad ; \quad i = -2,-3 \ . \tag{66}$$

Obviously, particular solutions of the inhomogeneous equation could then be found as superposition of particular solutions of the following equations:

$$\varepsilon_0 \frac{\partial^2 \breve{A}_i^{(I)}}{\partial t^2} + \varepsilon_0 c^2 k^2 \breve{A}_i^{(I)} = \breve{j}_{C,i}^{(I)} \quad ; \quad i = -2,-3 \ , \tag{67}$$

$$\varepsilon_0 \frac{\partial^2 \breve{A}_i^{(II)}}{\partial t^2} + \varepsilon_0 c^2 k^2 \breve{A}_i^{(II)} = \breve{j}_{C,i}^{(II)} \quad ; \quad i = -2,-3 \tag{68}$$

and of the general solution of

$$\varepsilon_0 \frac{\partial^2 \breve{A}_i^{(0)}}{\partial t^2} + \varepsilon_0 c^2 k^2 \breve{A}_i^{(0)} = 0 \quad ; \quad i = -2,-3 \ , \tag{69}$$

since addition of the last three differential equations results in equation (66) with

$$A_i(\vec{r},t) = A_i^{(0)}(\vec{r},t) + A_i^{(I)}(\vec{r},t) + A_i^{(II)}(\vec{r},t) \quad ; \quad i = -2,-3 \ . \tag{70}$$

For the following analysis, it is assumed that solutions $A_i^{(I)}$ and $A_i^{(II)}$ vanish, if the current densities vanish.

If it is supposed that the dynamics of the field in the second set are known and have not to be analyzed, then a similar argumentation as in sections 2.2 until 2.4 leads to the result that the wave equations for the first set alone could be reformulated as

$$\frac{\partial}{\partial t}\left(\partial L_{I,el} \Big/ \partial \dot{\breve{A}}_i^{(I)}\right) = \frac{\partial L_{I,el}}{\partial \breve{A}_i^{(I)}} \ ; \ \frac{\partial}{\partial t}\left(\partial L_{I,el} \Big/ \partial \left(\dot{\breve{A}}_i^{(I)}\right)^*\right) = \frac{\partial L_{I,el}}{\partial \left(\breve{A}_i^{(I)}\right)^*} \ ; \ k_1 > 0 \ , \tag{71}$$

with

$$L_{I,el} = \sum_{l_1,l_2,l_3=-\infty}^{\infty} \left\{ \frac{\varepsilon_0}{2}\left[\dot{\breve{\vec{A}}}^{(I)} \cdot \dot{\breve{\vec{A}}}^{(I)*} - c^2 \left(j\vec{k} \times \breve{\vec{A}}^{(I)}\right) \cdot \left(j\vec{k} \times \breve{\vec{A}}^{(I)}\right)^* \right] + \breve{\vec{A}}^{(I)} \cdot \left(\breve{\vec{j}}_C^{(I)}\right)^* + \left(\breve{\vec{A}}^{(I)}\right)^* \cdot \breve{\vec{j}}_C^{(I)} \right\} \tag{72}$$

or after application of Parseval's theorem

$$L_{I,el} = \int \frac{\varepsilon_0}{2}\left[\dot{\vec{A}}^{(I)} \cdot \dot{\vec{A}}^{(I)} - c^2 \left(\nabla \times \vec{A}^{(I)}\right) \cdot \left(\nabla \times \vec{A}^{(I)}\right) \right] d^3 r + \int \vec{A}^{(I)} \cdot \vec{j}_C^{(I)} \, d^3 r \ . \tag{73}$$

The situation of mechanical dynamics will be analyzed in the same way. We have

$$\phi(\vec{r},t) = \phi^{(I)}(\vec{r},t) + \phi^{(II)}(\vec{r},t) \tag{74}$$

with

$$\phi^{(I)}(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N_1}\frac{\Delta q_k}{|\vec{r}-\vec{r}_k|} \quad ; \quad \phi^{(II)}(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\sum_{k=N_1+1}^{N}\frac{\Delta q_k}{|\vec{r}-\vec{r}_k|} \quad . \tag{75}$$

The equations of motion for the first set of particles can then be derived by the Lagrange equations with the Lagrangian

$$\begin{aligned} L_{I,mech} &= -\frac{1}{8\pi\varepsilon_0}\sum_{\substack{m,n=1\\k\neq l}}^{N_1}\frac{\Delta q_m \Delta q_n}{|\vec{r}_m-\vec{r}_n|}-\sum_{n=1}^{N_1}\Delta q_n\,\phi^{(II)}(\vec{r}_n,t)+\sum_{n=1}^{N_1}\Delta m_n\frac{\dot{\vec{r}}_n\cdot\dot{\vec{r}}_n}{2} \\ &\quad +\int d^3r\left(\vec{A}^{(0)}(\vec{r},t)+\vec{A}^{(I)}(\vec{r},t)+\vec{A}^{(II)}(\vec{r},t)\right)\cdot\vec{j}_C^{(I)}(\vec{r},t) \\ &= -\frac{1}{8\pi\varepsilon_0}\sum_{\substack{m,n=1\\k\neq l}}^{N_1}\frac{\Delta q_m \Delta q_n}{|\vec{r}_m-\vec{r}_n|}-\sum_{n=1}^{N_1}\Delta q_n\,\phi^{(II)}(\vec{r}_n,t)+\sum_{n=1}^{N_1}\Delta m_n\frac{\dot{\vec{r}}_n\cdot\dot{\vec{r}}_n}{2} \\ &\quad +\sum_{n=1}^{N_1}\dot{\vec{r}}_n\,\Delta q_n\left(\vec{A}^{(0)}(\vec{r}_n,t)+\vec{A}^{(I)}(\vec{r}_n,t)+\vec{A}^{(II)}(\vec{r}_n,t)\right) \quad . \end{aligned} \tag{76}$$

Combining this function with the function $L_{I,el}$ from equation (72) yields a Lagrangian from which the Lagrange equations for set I of the charge carriers can be deduced:

$$\begin{aligned} L_I &= \sum_{l_1,l_2,l_3=-\infty}^{\infty}\frac{\varepsilon_0}{2}\left[\dot{\breve{\vec{A}}}^{(I)}\cdot\dot{\breve{\vec{A}}}^{(I)*}-c^2\left(j\vec{k}\times\breve{\vec{A}}^{(I)}\right)\cdot\left(j\vec{k}\times\breve{\vec{A}}^{(I)}\right)^*\right] \\ &\quad -\frac{1}{8\pi\varepsilon_0}\sum_{\substack{m,n=1\\m\neq n}}^{N_1}\frac{\Delta q_n \Delta q_m}{|\vec{r}_n-\vec{r}_m|}+\sum_{n=1}^{N_1}\Delta m_n\frac{\dot{\vec{r}}_n\cdot\dot{\vec{r}}_n}{2} \\ &\quad +\sum_{n=1}^{N_1}\Delta q_n\left[\vec{A}^{(II)}(\vec{r}_n,t)\cdot\dot{\vec{r}}_n+\sum_{l_1,l_2,l_3=-\infty}^{\infty}\left(\breve{\vec{A}}^{(0)}+\breve{\vec{A}}^{(I)}\right)e^{j\vec{k}\cdot\vec{r}_n}\cdot\dot{\vec{r}}_n-\phi^{(II)}(\vec{r}_n,t)\right] \end{aligned} \tag{77}$$

Note that in the first sum only field terms are found that are generated from the first set of particles.

Would we prefer to obtain a Lagrangian that additionally yields the solution for the free space, then one possible solution were

$$L_{0,I}=L_I+\sum_{l_1,l_2,l_3=-\infty}^{\infty}\frac{\varepsilon_0}{2}\left[\dot{\breve{\vec{A}}}^{(0)}\cdot\dot{\breve{\vec{A}}}^{(0)*}-c^2\left(j\vec{k}\times\breve{\vec{A}}^{(0)}\right)\cdot\left(j\vec{k}\times\breve{\vec{A}}^{(0)}\right)^*\right] \tag{78}$$

A quite similar Lagrangian has already been derived in literature, e.g. [7], [8], the main difference is concerning the Coulomb self-energy, the splitting into terms that stem from free space and from the impedance, and/or the formulation using Fourier series instead of Fourier transforms.

4 The classical Hamiltonian for a system that is controlled by external forces

From the above classical Lagrangian of a system that is controlled by external forces, we will systematically derive a Hamiltonian.

4.1 Canonical momenta

First of all, the canonical momenta are found as

$$p_{n,i} = \frac{\partial L}{\partial \dot{r}_{n,i}}$$
$$= \Delta m_n \dot{r}_{n,i} + \Delta q_n \left(A_i^{(0)}(\vec{r}_n) + A_i^{(I)}(\vec{r}_n) + A_i^{(II)}(\vec{r}_n, t) \right) \;;\; n = 1, \dots N_1 \;;\; i = 1,2,3 \;, \quad (79)$$

$$\breve{\Pi}_i^{(I)} = \frac{\partial L}{\partial \dot{\breve{A}}_i^{(I)*}} = \varepsilon_0 \dot{\breve{A}}_i^{(I)} \quad ; \quad \breve{\Pi}_i^{(I)*} = \frac{\partial L}{\partial \dot{\breve{A}}_i^{(I)}} = \varepsilon_0 \dot{\breve{A}}_i^{(I)*} \quad ; \quad i = -2, -3 \;, \qquad (80)$$

$$\breve{\Pi}_i^{(0)} = \frac{\partial L}{\partial \dot{\breve{A}}_i^{(0)*}} = \varepsilon_0 \dot{\breve{A}}_i^{(0)} \quad ; \quad \breve{\Pi}_i^{(0)*} = \frac{\partial L}{\partial \dot{\breve{A}}_i^{(0)}} = \varepsilon_0 \dot{\breve{A}}_i^{(0)*} \quad ; \quad i = -2, -3 \qquad (81)$$

4.2 The classical Hamiltonian

We define vectors

$$\vec{\breve{\Pi}}^{(I)} = \left(0, \breve{\Pi}_{-2}^{(I)}, \breve{\Pi}_{-3}^{(I)}\right)^T \;, \qquad (82)$$

$$\vec{\breve{\Pi}}^{(0)} = \left(0, \breve{\Pi}_{-2}^{(0)}, \breve{\Pi}_{-3}^{(0)}\right)^T \qquad (83)$$

for each value of (l_1, l_2, l_3). The corresponding classical Hamiltonian is then found by the following relation (see for example [6]):

$$H = \sum_{n=1}^{N_1} \vec{p}_n \cdot \dot{\vec{r}}_n - L$$
$$+ \sum_{\substack{l_1>0,\\ l_2, l_3=-\infty}}^{\infty} \left(\vec{\breve{\Pi}}^{(0)*} \cdot \dot{\vec{\breve{A}}}^{(0)} + \vec{\breve{\Pi}}^{(0)} \cdot \dot{\vec{\breve{A}}}^{(0)*} + \vec{\breve{\Pi}}^{(I)*} \cdot \dot{\vec{\breve{A}}}^{(I)} + \vec{\breve{\Pi}}^{(I)} \cdot \dot{\vec{\breve{A}}}^{(I)*} \right)$$

$$H = \sum_{n=1}^{N_1} \vec{p}_n \cdot \dot{\vec{r}}_n - L$$

$$+\frac{1}{2} \sum_{l_1,l_2,l_3=-\infty}^{\infty} \left(\breve{\vec{\Pi}}^{(0)*} \cdot \dot{\breve{\vec{A}}}^{(0)} + \breve{\vec{\Pi}}^{(0)} \cdot \dot{\breve{\vec{A}}}^{(0)*} + \breve{\vec{\Pi}}^{(I)*} \cdot \dot{\breve{\vec{A}}}^{(I)} + \breve{\vec{\Pi}}^{(I)} \cdot \dot{\breve{\vec{A}}}^{(I)*} \right) . \tag{84}$$

Inserting the Lagrangian from equation (78) in combination with equations (77) yields

$$\begin{aligned} H = & \sum_{l_1,l_2,l_3=-\infty}^{\infty} \left\{ \frac{1}{2\,\varepsilon_0} \breve{\vec{\Pi}}^{(0)*} \cdot \breve{\vec{\Pi}}^{(0)} + \frac{\varepsilon_0}{2} c^2 \left(j\vec{k} \times \breve{\vec{A}}^{(0)} \right) \cdot \left(j\vec{k} \times \breve{\vec{A}}^{(0)} \right)^* \right\} \\ & + \sum_{l_1,l_2,l_3=-\infty}^{\infty} \left\{ \frac{1}{2\,\varepsilon_0} \breve{\vec{\Pi}}^{(I)*} \cdot \breve{\vec{\Pi}}^{(I)} + \frac{\varepsilon_0}{2} c^2 \left(j\vec{k} \times \breve{\vec{A}}^{(I)} \right) \cdot \left(j\vec{k} \times \breve{\vec{A}}^{(I)} \right)^* \right\} \\ & + \sum_{n=1}^{N_1} \frac{1}{2\,\Delta m_n} \left\{ \vec{p}_n - \Delta q_n \left(\sum_{l_1,l_2,l_3=-\infty}^{\infty} \left(\breve{\vec{A}}^{(0)} + \breve{\vec{A}}^{(I)} \right) e^{j\vec{k}\cdot\vec{r}_n} + \vec{A}^{(II)}\left(\vec{r}_n, t\right) \right) \right\}^2 \\ & + \frac{1}{8\pi\varepsilon_0} \sum_{\substack{m,n=1 \\ n\neq l}}^{N_1} \frac{\Delta q_m \Delta q_n}{\left|\vec{r}_m - \vec{r}_n\right|} + \sum_{n=1}^{N_1} \Delta q_n \phi^{(II)}\left(\vec{r}_n, t\right) \end{aligned} \tag{85}$$

If it is assumed for a moment that all potentials are zero, then there remains a Hamiltonian

$$H_{kin} = \sum_{n=1}^{N_1} \frac{1}{2\,\Delta m_n} \vec{p}_n^{\,2} \tag{86}$$

It has the dimension of energy and it represents the kinetic energy of all particles in set *I*. All other sum terms in *H* must therefore also have the dimension of energy. The first two sums in equation (85) describe the energy in the radiation field that does not come from set *II*, the last two sums indicate the Coulomb energy. The remaining part describes the interaction between the particles of set *I* and the field terms.

4.3 Reformulation by means of eigenfunctions

In section 2.2, it was shown that substitution of the –2nd and –3rd component of the vector potential and its derivatives by

$$\breve{X}_i = +j \frac{\varepsilon_0\, c\, k}{2} \breve{A}_i + \frac{1}{2} \breve{\Pi}_i \quad ; \quad i = -2, -3 \tag{87}$$

$$\breve{Y}_i = -j \frac{\varepsilon_0\, c\, k}{2} \breve{A}_i + \frac{1}{2} \breve{\Pi}_i \quad ; \quad i = -2, -3 \tag{88}$$

led to an advantageous formulation. We will try to introduce this formulation in the above Hamiltonian density. Using

$$\breve{A}_i^{(I)} = \frac{1}{j\,\varepsilon_0\,c\,k}\left(\breve{X}_i^{(I)} - \breve{Y}_i^{(I)}\right) \;\; ; \;\; \breve{A}_i^{(0)} = \frac{1}{j\,\varepsilon_0\,c\,k}\left(\breve{X}_i^{(0)} - \breve{Y}_i^{(0)}\right) \;\; ; \;\; i = -2, -3 \;, \tag{89}$$

$$\breve{\Pi}_i^{(I)} = \breve{X}_i^{(I)} + \breve{Y}_i^{(I)} \quad ; \quad \breve{\Pi}_i^{(0)} = \breve{X}_i^{(0)} + \breve{Y}_i^{(0)} \quad ; \quad i = -2, -3 \;, \tag{90}$$

$$\breve{\vec{X}}^{(v)} = \left(0, \breve{X}_{-2}^{(v)}, \breve{X}_{-3}^{(v)}\right)^T \quad ; \quad \breve{\vec{Y}}^{(v)} = \left(0, \breve{Y}_{-2}^{(v)}, \breve{Y}_{-3}^{(v)}\right)^T \quad ; \quad v = 0, I \tag{91}$$

and of

$$\breve{X}_i^{(v)*}\left(-\vec{k}, t\right) = -j\frac{\varepsilon_0\,c\,k}{2}\breve{A}_i^{(v)}\left(+\vec{k}, t\right) + \frac{1}{2}\breve{\Pi}_i^{(v)}\left(+\vec{k}, t\right) = \breve{Y}_i^{(v)}\left(+\vec{k}, t\right) \;\; ; \;\; i = -2, -3 \tag{92}$$

$$\breve{Y}_i^{(v)*}\left(-\vec{k}, t\right) = +j\frac{\varepsilon_0\,c\,k}{2}\breve{A}_i^{(v)}\left(+\vec{k}, t\right) + \frac{1}{2}\breve{\Pi}_i^{(v)}\left(+\vec{k}, t\right) = \breve{X}_i^{(v)}\left(+\vec{k}, t\right) \;\; ; \;\; i = -2, -3 \tag{93}$$

leads to

$$\begin{aligned} H = {} & \frac{1}{\varepsilon_0} \sum_{l_1, l_2, l_3 = -\infty}^{\infty} \left\{ \breve{\vec{Y}}^{(0)*} \cdot \breve{\vec{Y}}^{(0)} + \breve{\vec{Y}}^{(0)} \cdot \breve{\vec{Y}}^{(0)*} + \breve{\vec{Y}}^{(I)*} \cdot \breve{\vec{Y}}^{(I)} + \breve{\vec{Y}}^{(I)} \cdot \breve{\vec{Y}}^{(I)*} \right\} \\ & + \sum_{n=1}^{N_1} \frac{1}{2\,\Delta m_n} \left\{ \vec{p}_n - \Delta q_n \left(\vec{A}^{(II)}\left(\vec{r}_n, t\right) + \sum_{l_1, l_2, l_3 = -\infty}^{\infty} \frac{\breve{\vec{X}}^{(0)} - \breve{\vec{Y}}^{(0)} + \breve{\vec{X}}^{(I)} - \breve{\vec{Y}}^{(I)}}{j\,\varepsilon_0\,c\,k} e^{j\vec{k}\cdot\vec{r}_n} \right) \right\}^2 \\ & + \frac{1}{8\pi\varepsilon_0} \sum_{\substack{m,n=1 \\ n \neq l}}^{N_1} \frac{\Delta q_m \Delta q_n}{\left|\vec{r}_m - \vec{r}_n\right|} + \sum_{n=1}^{N_1} \Delta q_n \phi^{(II)}\left(\vec{r}_n, t\right) \end{aligned} \tag{94}$$

This is indeed a somewhat simpler expression as compared to the one that was used earlier because of the missing cross-products. However, the new vectors are not canonical variables. Therefore, they are not so well suited for deriving equations of motion. Anyway, it will be seen later that they are very useful in finding suitable operators for the quantum case.

5 The Quantum-Hamiltonian

5.1 Canonical quantization

Having found the classical Hamiltonian, it is now easy to perform the transition to a quantum-electrodynamical description. We have simply to interpret all canonical coordinates, all canonical momenta and all expressions derived from them as operators instead of functions. Furthermore, the operators of particle coordinates and their conjugate momenta have to meet the following commutation relations in the Schrödinger picture:

$$\left[r_{n,i}, p_{m,l}\right] = j\hbar\,\delta_{i,l}\,\delta_{n,m} \;\;;\;\; i,l = 1,2,3 \;\;;\;\; n,m = 1,\ldots,N_1 \;\;, \tag{95}$$

$$\left[\breve{A}_i^{(m)}\left(\vec{k},t\right), \breve{\Pi}_l^{(n)\dagger}\left(\vec{k}',t\right)\right] = j\hbar\,\delta_{i,l}\,\delta_{m,n}\delta_{\vec{k},\vec{k}'} \;\;;\;\; i,l = -2,-3 \;\;;\;\; m = 0,I \;\;, \tag{96}$$

$$\left[\breve{A}_i^{(m)\dagger}\left(\vec{k},t\right), \breve{\Pi}_l^{(n)}\left(\vec{k}',t\right)\right] = j\hbar\,\delta_{i,l}\,\delta_{m,n}\delta_{\vec{k},\vec{k}'} \;\;;\;\; i,l = -2,-3 \;\;;\;\; m = 0,I \;\;. \tag{97}$$

$\breve{\Pi}_l^{\dagger}$ denotes the operator that is adjoint to $\breve{\Pi}_l$ and so on. All other commutators are zero. The above procedure is known as *canonical quantization*.

5.2 Creation and annihilation operators

In chapter 2, the eigenfunctions of the wave equation were considered as possible replacements of the fields and their conjugate momenta. In the same way, field operators and their conjugate momentum operators are now considered. We will therefore analyze the operators

$$\breve{X}_i^{(v)}\left(\vec{k},t\right) = +j\frac{\varepsilon_0 c k}{2}\breve{A}_i^{(v)}\left(\vec{k},t\right) + \frac{1}{2}\breve{\Pi}_i^{(v)}\left(\vec{k},t\right) \;\;;\;\; v = 0,I \;\;;\;\; i = -2,-3 \tag{98}$$

$$\breve{Y}_i^{(v)}\left(\vec{k},t\right) = -j\frac{\varepsilon_0 c k}{2}\breve{A}_i^{(v)}\left(\vec{k},t\right) + \frac{1}{2}\breve{\Pi}_i^{(v)}\left(\vec{k},t\right) \;\;;\;\; v = 0,I \;\;;\;\; i = -2,-3 \tag{99}$$

$$\breve{X}_i^{(v)\dagger}\left(-\vec{k},t\right) = \breve{Y}_i^{(v)}\left(+\vec{k},t\right) \;\;;\;\; v = 0,I \;\;;\;\; i = -2,-3 \tag{100}$$

where again $\breve{A}_i^{(v)}$ and $\breve{\Pi}_i^{(v)}$ are considered to be operators. Obviously, it follows

$$\begin{aligned}
&\left[\breve{X}_i^{(v)}\left(\vec{k},t\right), \breve{X}_i^{(\mu)}\left(\vec{k}',t\right)\right] = \left[\breve{Y}_i^{(v)}\left(\vec{k},t\right), \breve{Y}_i^{(\mu)}\left(\vec{k}',t\right)\right] = 0 \;\;;\;\; v,\mu = 0,I \;\;;\;\; i = -2,-3 \\
&\left[\breve{X}_i^{(v)}\left(\vec{k},t\right), \breve{Y}_i^{(\mu)}\left(\vec{k}',t\right)\right] = 0 \;\;;\;\; v,\mu = 0,I \;\;;\;\; i = -2,-3
\end{aligned} \tag{101}$$

and without any restriction with respect to the values of k_{-1} or k'_{-1}. Furthermore, it is

$$\left[\breve{X}_i^{(v)}\left(\vec{k},t\right), \breve{X}_l^{(\mu)\dagger}\left(\vec{k}',t\right)\right] = +\frac{\hbar}{2}\varepsilon_0\, c\, k\,\delta_{i,l}\,\delta_{v,\mu}\,\delta_{\vec{k},\vec{k}'} \;\;;\;\; i,l = -2,-3 \tag{102}$$

$$\left[\breve{Y}_i^{(v)}\left(\vec{k},t\right), \breve{Y}_l^{(\mu)\dagger}\left(\vec{k}',t\right)\right] = -\frac{\hbar}{2}\varepsilon_0\, c\, k\,\delta_{i,l}\,\delta_{v,\mu}\,\delta_{\vec{k},\vec{k}'} \;\;;\;\; i,l = -2,-3 \tag{103}$$

A simpler relation could thus be found by introducing a new operator

$$\breve{a}_i^{(v)}\left(\vec{k}\right) := \frac{-\breve{Y}_i^{(v)}\left(\vec{k}\right)}{\sqrt{-\hbar\,\varepsilon_0\, c\, k/2}} \tag{104}$$

or

$$\breve{a}_i^{(\nu)}\left(\vec{k}\right) := \frac{\varepsilon_0\, c\, k\ \breve{A}_i^{(\nu)}\left(\vec{k},t\right) + j\ \breve{\Pi}_i^{(\nu)}\left(\vec{k},t\right)}{\sqrt{2\hbar\ \varepsilon_0\, c\, k}} \ ;\ \nu = 0, I\ ;\ i = -2,-3 \tag{105}$$

from which we have

$$\breve{a}_i^{(\nu)\dagger}\left(\vec{k}\right) = \frac{\varepsilon_0\, c\, k\ \breve{A}_i^{(\nu)\dagger}\left(\vec{k},t\right) - j\ \breve{\Pi}_i^{(\nu)\dagger}\left(\vec{k},t\right)}{\sqrt{2\hbar\ \varepsilon_0\, c\, k}} \ ;\ \nu = 0, I\ ;\ i = -2,-3 \tag{106}$$

and thus

$$\left[\breve{a}_i^{(\nu)}\left(\vec{k},t\right), \breve{a}_l^{(\mu)\dagger}\left(\vec{k}',t\right)\right] = \delta_{i,l}\delta_{\nu,\mu}\delta_{\vec{k},\vec{k}'} \ ;\ \nu,\mu = 0, I\ ;\ i,l = -2,-3\ . \tag{107}$$

The a- and $a^\dagger$- operators are the well-known as annihilation and creation operators.

It is emphasized that the two components of the newly defined vector operator are linked by the fact that they depend on the same propagation vector $\vec{k}$ and that in particular the coefficients in the defining equations (104) and (106) depend on the absolute value k of $\vec{k}$. Otherwise, commutation relation (107) would not hold. We will see that this again will generate an important distinction as compared to the explanations given by Cohen-Tannoudji, Dupont-Roc and Grynberg in [7].

5.3 The Hamiltonian

From the above definitions, it follows

$$\breve{A}_i^{(\nu)}\left(\vec{k},t\right) = \sqrt{\frac{\hbar}{2}\frac{1}{\varepsilon_0\, c\, k}}\left\{\breve{a}_i^{(\nu)}\left(\vec{k}\right) + \breve{a}_i^{(\nu)\dagger}\left(-\vec{k}\right)\right\}\ , \tag{108}$$

$$\breve{\Pi}_i^{(\nu)}\left(\vec{k},t\right) = \frac{1}{j}\sqrt{\frac{\hbar}{2}\varepsilon_0\, c\, k}\left\{\breve{a}_i^{(\nu)}\left(\vec{k}\right) - \breve{a}_i^{(\nu)\dagger}\left(-\vec{k}\right)\right\}\ . \tag{109}$$

Using the newly defined operators, we find for the quantum Hamiltonian for a system that is controlled by external forces:

$$\begin{aligned} H = {} & \frac{\hbar}{2}\sum_{l_1,l_2,l_3=-\infty}^{\infty} c\,k\left\{\breve{a}_{-2}^{(0)\dagger}\breve{a}_{-2}^{(0)} + \breve{a}_{-2}^{(0)}\breve{a}_{-2}^{(0)\dagger} + \breve{a}_{-3}^{(0)\dagger}\breve{a}_{-3}^{(0)} + \breve{a}_{-3}^{(0)}\breve{a}_{-3}^{(0)\dagger}\right\} \\ & + \frac{\hbar}{2}\sum_{l_1,l_2,l_3=-\infty}^{\infty} c\,k\left\{\breve{a}_{-2}^{(I)\dagger}\breve{a}_{-2}^{(I)} + \breve{a}_{-2}^{(I)}\breve{a}_{-2}^{(I)\dagger} + \breve{a}_{-3}^{(I)\dagger}\breve{a}_{-3}^{(I)} + \breve{a}_{-3}^{(I)}\breve{a}_{-3}^{(I)\dagger}\right\} \\ & + \sum_{n=1}^{N_1}\frac{1}{2\,\Delta m_n}\left\{\vec{p}_n - \Delta q_n\left(\vec{A}^{(0)}\left(\vec{r}_n,t\right) + \vec{A}^{(I)}\left(\vec{r}_n,t\right) + \vec{A}^{(II)}\left(\vec{r}_n,t\right)\right)\right\}^2 \\ & + \frac{1}{8\pi\varepsilon_0}\sum_{\substack{m,n=1\\ n\neq l}}^{N_1}\frac{\Delta q_m \Delta q_n}{\left|\vec{r}_m - \vec{r}_n\right|} + \sum_{n=1}^{N_1}\Delta q_n \phi^{(II)}\left(\vec{r}_n,t\right) \qquad . \end{aligned} \tag{110}$$

5.4 Photons

If for a moment a system is considered that does not contain any particles, then the Hamiltonian reduces to

$$H = \frac{\hbar}{2} \sum_{l_1,l_2,l_3=-\infty}^{\infty} c\,k \left\{ -\breve{a}_{-2}^{(0)\dagger} \breve{a}_{-2}^{(0)} - \breve{\breve{a}}_{-2}^{(0)} \breve{\breve{a}}_{-2}^{(0)\dagger} - \breve{a}_{-3}^{(0)\dagger} \breve{a}_{-3}^{(0)} - \breve{\breve{a}}_{-3}^{(0)} \breve{\breve{a}}_{-3}^{(0)\dagger} \right\} \ . \tag{111}$$

Since no particles are included, only quantized radiation will be described. That means: This Hamiltonian describes photons.

Application of the commutation relations yields

$$H = \sum_{l_1,l_2,l_3=-\infty}^{\infty} \hbar\, c\, k \left\{ \breve{a}_{-2}^{(0)\dagger} \breve{a}_{-2}^{(0)} + \breve{a}_{-3}^{(0)\dagger} \breve{a}_{-3}^{(0)} + 1 \right\} \ . \tag{112}$$

Now let us consider the sum term

$$H_{l_1,l_2,l_3} = \hbar\, c\, k \left\{ \breve{a}_{-2}^{(0)\dagger} \breve{a}_{-2}^{(0)} + \breve{a}_{-3}^{(0)\dagger} \breve{a}_{-3}^{(0)} + 1 \right\} \quad ; \quad k = \frac{2\pi}{L} \sqrt{l_1^2 + l_2^2 + l_3^2} \ . \tag{113}$$

This is a Hamiltonian that describes a 2-dimensional harmonic quantum oscillator the energy-eigenvalues of which are non-negative integer multiples of $\hbar ck$. The dimensionality stems from the fact that electromagnetic waves are transverse waves with two orthogonal polarization directions.

Now where we have identified H_{l_1,l_2,l_3} as operator for a harmonic oscillator, it is sensible to introduce the angular frequency of this oscillator:

$$\omega(k) = c\,k \ . \tag{114}$$

The eigenvalues of the number operators

$$N_{-2}\left(\vec{k}\right) := \breve{a}_{-2}^{(0)\dagger} \breve{a}_{-2}^{(0)} \quad ; \quad N_{-3}\left(\vec{k}\right) := \breve{a}_{-3}^{(0)\dagger} \breve{a}_{-3}^{(0)} \tag{115}$$

will be called $n_{-2}\left(\vec{k}\right)$ and $n_{-3}\left(\vec{k}\right)$. Since the least eigenvalues that can be accepted are 0, it follows that the least energy-eigenvalue that corresponds to a *non-vanishing* eigenvector is

$$E_{\min}\left(\vec{k}\right) = \hbar\omega \ . \tag{116}$$

We conclude from this that the number of photons that occupy a wave with wave-vector $\vec{k}$, is given by

$$n\left(\vec{k}\right) := \begin{cases} n_{-2}\left(\vec{k}\right) + n_{-3}\left(\vec{k}\right) + 1, & \text{if a non-vanishing eigenvector exists} \\ 0 & \text{otherwise} \end{cases} \ . \tag{117}$$

Therefore, there is no need to talk about “zero-point energy” of a photon: Either a photon exists, then it transports energy $\hbar\omega$, or there is no photon and no energy is transported.

5.5 Comparison with other Hamiltonians used to describe systems with noise

We are now able to compare the Hamiltonian derived in equation (110) with other Hamiltonians that were used in literature to describe systems that produce noise. For convenience, we will reformulate the newly derived Hamiltonian first:

$$H = H^{(S)} + H_R^{(0)} + H_I^{(A)} + H_I^{(V)} \tag{118}$$

with

$$H_R^{(0)} = \sum_{l_1,l_2,l_3=-\infty}^{\infty} \hbar\omega \left\{ \breve{a}_{-2}^{(0)\dagger} \breve{a}_{-2}^{(0)} + \breve{a}_{-3}^{(0)\dagger} \breve{a}_{-3}^{(0)} + 1 \right\} , \tag{119}$$

$$\begin{aligned} H^{(I)} = &\sum_{l_1,l_2,l_3=-\infty}^{\infty} \hbar\omega \left\{ \breve{a}_{-2}^{(I)\dagger} \breve{a}_{-2}^{(I)} + \breve{a}_{-3}^{(I)\dagger} \breve{a}_{-3}^{(I)} + 1 \right\} + \frac{1}{8\pi\varepsilon_0} \sum_{\substack{m,n=1 \\ n\neq l}}^{N_1} \frac{\Delta q_m \Delta q_n}{\left|\vec{r}_m - \vec{r}_n\right|} \\ &+ \sum_{n=1}^{N_1} \frac{1}{2\,\Delta m_n} \left\{ \vec{p}_n - \Delta q_n \vec{A}^{(I)}\left(\vec{r}_n, t\right) \right\}^2 \quad , \end{aligned} \tag{120}$$

$$H_I^{(V)} = \sum_{n=1}^{N_1} \Delta q_n \phi^{(II)}\left(\vec{r}_n, t\right) , \tag{121}$$

$$\begin{aligned} H_I^{(A)} = &\sum_{n=1}^{N_1} \frac{1}{2\Delta m_n} \left\{ \vec{p}_n - \Delta q_n \vec{A}^{(I)}\left(\vec{r}_n, t\right) \right\} \left\{ \Delta q_n \vec{A}^{(0)}\left(\vec{r}_n, t\right) + \Delta q_n \vec{A}^{(II)}\left(\vec{r}_n, t\right) \right\} \\ &+ \sum_{n=1}^{N_1} \frac{1}{2\Delta m_n} \left\{ \Delta q_n \vec{A}^{(0)}\left(\vec{r}_n, t\right) + \Delta q_n \vec{A}^{(II)}\left(\vec{r}_n, t\right) \right\} \left\{ \vec{p}_n - \Delta q_n \vec{A}^{(I)}\left(\vec{r}_n, t\right) \right\} \\ &+ \sum_{n=1}^{N_1} \frac{1}{2\,\Delta m_n} \left\{ \Delta q_n \vec{A}^{(0)}\left(\vec{r}_n, t\right) + \Delta q_n \vec{A}^{(II)}\left(\vec{r}_n, t\right) \right\}^2 \quad . \end{aligned} \tag{122}$$

That is: we can decompose the Hamiltonian into four parts:

- $H_R^{(0)}$ that describes free photons,
- $H^{(I)}$ that describes particle-set *I* (the noise producing system), without any disturbance by external influences
- $H_I^{(V)}$ that describes the interaction of set *I* with external scalar potentials,
- and $H_I^{(A)}$ that describes the interaction of set *I* with external vector-potentials

Scanning the literature yields the surprising result that most authors do not take into consideration the interaction term with the vector-potential.

Grau and Kleen [10], for instance, use a Hamiltonian that is written as

$$H = H^{(S)} + \sum_{\alpha} \hbar\omega_{\alpha}\left(a_{\alpha}^{\dagger}a_{\alpha} + \frac{1}{2}\right) . \tag{123}$$

With goodwill, the second term may be interpreted as an operator that describes free photons. However, in the context of their paper, it turns out that they do not recognize the two-dimensional character of the oscillations nor do they take into consideration any interaction process.

Kubo [5] uses the Hamiltonian

$$H = H^{(S)} + \sum_{i} V_i(t) Q_i(\ldots,\vec{r}_{\alpha},\ldots,\vec{p}_{\alpha},\ldots) . \tag{124}$$

Since the arguments of Q_i in this formula are supposed to be canonical coordinates and momenta, this equation could be interpreted as a (very general) formulation of the newly found Hamiltonian. However, in the context of Kubo's book, it turns out that V_i means an external voltage. Therefore, the second term in Kubo's Hamiltonian corresponds to $H_I^{(V)}$. Thus, no interaction with the vector potential is taken into account.

Both examples show that essential parts of the Hamiltonian are missing.

There are publications where more detailed Hamiltonians are used, e.g. in publications by van Vliet and Handel [2–4], [11].

5.6 Reformulation

In the interaction parts of the Hamiltonian an in one sum term of the part that describes the impedance, terms including $\vec{A}^{(0)}(\vec{r}_n,t)$ and $\vec{A}^{(I)}(\vec{r}_n,t)$ appear. We have not yet replaced them by terms of annihilation and creation operators for conciseness. However, it will be easier to handle the Hamiltonian, if also these terms are replaced. We take into consideration that

$$\vec{A}(\vec{r},t) = \frac{1}{2}\sum_{l_1,l_2,l_3=-\infty}^{\infty}\left\{\breve{\vec{A}}(\vec{k},t)e^{j\vec{k}\cdot\vec{r}} + e^{j\vec{k}\cdot\vec{r}}\breve{\vec{A}}(\vec{k},t)\right\} \;\; ; \;\; \vec{k} = \frac{2\pi}{L_{cube}}(l_1\vec{e}_1 + l_2\vec{e}_2 + l_3\vec{e}_3) . \tag{125}$$

Inserting equations (108) and (114) it follows

$$\vec{A}^{(v)}(\vec{r},t) = \frac{1}{2}\sum_{\vec{k}}^{\infty}\sqrt{\frac{\hbar}{2}\frac{1}{\varepsilon_0\,\omega}}\left\{\left[\breve{a}_i^{(v)}(\vec{k}) + \breve{a}_i^{(v)\dagger}(-\vec{k})\right]e^{j\vec{k}\cdot\vec{r}} + e^{j\vec{k}\cdot\vec{r}}\left[\breve{a}_i^{(v)}(\vec{k}) + \breve{a}_i^{(v)\dagger}(-\vec{k})\right]\right\} . \tag{126}$$

Note that in this expression a term $1/\omega(k)^{1/2}$ appears for every sum term. Note furthermore that in the interaction Hamiltonian $H_I^{(A)}$ and in the system Hamiltonian $H^{(I)}$ the square of these operators is found. That means: We have terms where obviously a factor $1/f$ occurs!

Terms similar to these were already used by C. van Vliet [11] in 1990 as the author learned during the VIIIth van der Ziel Symposium. There, a deduction of $1/f$-noise for certain processes was already derived.

5 Conclusions

In this paper, a QED-Hamiltonian was derived rigorously within the range of a model that did only include point masses, point charges, and that did not include other effects than those described by the laws of electrodynamics. It was shown that by proper formulation of the equations, diverging energy-terms (self-energy and energy of photons) can be avoided.

Models that are in use to explain thermal noise (Grau-Kleen and Kubo) were recognized to be incomplete. A link to a quite similar description used by C. van Vliet already ten years ago, was seen during the viii-th van der Ziel symposium where this paper was presented.

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