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A novel approach for teaching electrodynamics

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***Abstract*-- In electronics and information engineering (EIE), teaching Maxwell-theory is always a challenge, since this theory accumulates mathematical and physics problems at the same time. As a result, students will see mainly the mathematical problems of the theory. A deeper insight into the physics behind the theory will be made difficult. Consequently, only few EIE students will show the necessary comprehension of this material. In this paper, a mathematical method based on a fourfold Fourier transform is introduced that makes use of the improved mathematical education that EIE students nowadays receive, particularly in signal processing. It maps partial differential equations completely into simple algebraic equations, and these can be solved easily. As a result of this method, the lecturer is enabled to concentrate more on physics and engineering instead of mathematical difficulties. Students will thus gain a better comprehension of the material. The method will be described in more detail. Some examples will be given to demonstrate its usefulness and to show its didactic benefits.**

***Index Terms*—Electrical Engineering Education, Electrodynamics, Transforms, Wave Equations**

I. Introduction

Lectures on electrodynamics have always been a challenge, not only for students but also for lecturers, since the subject combines several mathematical and physical difficulties at the same time. Thus, it happens frequently that students only see the mathematical problems without getting a deeper understanding of the physics behind them.

Therefore, a procedure is proposed that reduces mathematical problems. The price for this simplification is the introduction of a mathematical tool that normally is only used in a less complex form. This tool is the application of a combination of a Fourier transform from the domain of time-domain functions to the range of frequency-domain functions and of a triple Fourier transform from the domain of ordinary-space-domain functions to the range of reciprocal-space-domain functions. It appears, however, as if this price was not too high, since the application of Fourier transforms is nowadays an everyday job even for undergraduate students in their last year.

Why should this procedure be advantageous? It is because it transforms a very abstract description using partial differential equations completely into a more tangible geometrical description. It thus stimulates the powers of imagination and opens the mind for interpretation instead of establishing an attitude of mechanically applying mathematical algorithms. This procedure shall be outlined in the following.

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II. Generalized Fourier transforms

A. *The ordinary Fourier transform as a replacement for phasor notation*

The starting point for the novel approach is the thesis that the ordinary Fourier transform is needed anyway in solving Maxwell's equations in a – maybe lossy – dielectric medium. This contention might be surprising for someone or the other, but constitutional relations use to be frequency dependent. I.e.: relative permittivity and relative permeability are transfer functions in the sense of signal processing. That means they must be Fourier transforms of impulse responses, linking the Fourier transform of one physical quantity to the other.

It could thus be a good idea to interpret the relation between electric field and dielectric flux density and the relation between magnetic induction and magnetic field as relations between fields in the frequency domain. Indeed, since even sophomore students nowadays are relatively familiar with the ordinary Fourier transform, there should be no problem with this interpretation.

Thus, if $f(t)$ is a time-domain function with $\hat{f}(j\omega)$ being its ordinary Fourier transform,

$$\hat{f}(j\omega)=\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt=\boldsymbol{\mathcal{F}}_{t\to\omega}\{f\}, \tag{1}$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(j\omega)\, e^{j\omega t}\, d\omega=\boldsymbol{\mathcal{F}}_{t\leftarrow\omega}^{-1}\{\hat{f}\}, \tag{2}$$

then the constitutive relations could be written as follows:

$$\hat{\vec{B}}(j\omega,\vec{r})=\hat{\mu}(j\omega,\vec{r})\,\hat{\vec{H}}(j\omega,\vec{r}), \tag{3}$$

$$\hat{\vec{D}}(j\omega,\vec{r})=\hat{\varepsilon}(j\omega,\vec{r})\,\hat{\vec{E}}(j\omega,\vec{r}), \tag{4}$$

where $\vec{E},\vec{D},\vec{B}$ and $\vec{H}$ (all with arguments $(t,\vec{r})$) are the electric field, the electric flux density, the magnetic induction and the magnetic field in the normal space-time domain, while the quantities marked with a "hat"-symbol are their respective Fourier transforms in the space-frequency domain. $\hat{\varepsilon}$ and $\hat{\mu}$ are the (complex) permittivity and the permeability in the space-frequency domain.

For the rest of this paper, it will be assumed, that permittivity and permeability do not depend on position in order to simplify the situation and to work out the essentials of the new method. It goes without saying that the method is also advantageously applicable for linear media with position-dependent media.

The benefit of the formulation using ordinary Fourier transforms is the applicability of the theorems that are known from the theory of these transforms, e.g. a derivation with respect to time is mapped into a multiplication by $j\omega$, or a shift in time is mapped into a multiplication by a simple complex factor, which makes mathematics easier.

As a matter of fact, many lecturers use this procedure for teaching electrodynamics with the only difference that they prefer to interpret relations as phasor relations (see for instance [1]).

B. The Fourier spatial transform

Another approach for solving Maxwell's equations can be used for considering electrodynamics in vacuum. Since then permittivity and permeability are simple factors of proportionality, ordinary Fourier transform has not necessarily to be used. The main idea, though, is akin to that described above: in order to get rid of derivatives, Fourier transforms are applied. However, because this time derivatives with respect to three independent *position variables* are to be eliminated, a three-fold Fourier transform has to be applied and that is the Fourier spatial transform.

The Fourier spatial transform from a function $f(r_1, r_2, r_3)$ to a function $\widehat{f}(k_1, k_2, k_3)$ is defined as

$$\widehat{f}(k_1,k_2,k_3) = \boldsymbol{\mathcal{F}}_{(r_1,r_2,r_3)\to(k_1,k_2,k_3)}\left\{f(r_1,r_2,r_3)\right\} = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(r_1,r_2,r_3)\, e^{-jk_1r_1}\, dr_1\right] e^{-jk_2r_2}\, dr_2\right\} e^{-jk_3r_3}\, dr_3 \,. \quad (5)$$

Using an arbitrary orthogonal basis of unit vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in ordinary space and with the abbreviation

$$\vec{r} = (r_1, r_2, r_3)^T = r_1\,\vec{e}_1 + r_2\,\vec{e}_2 + r_3\,\vec{e}_3 \,, \quad (6)$$

$$\vec{k} = (k_1, k_2, k_3)^T = k_1\,\vec{e}_1 + k_2\,\vec{e}_2 + k_3\,\vec{e}_3 \,, \quad (7)$$

a more concise notation is obtained:

$$\widehat{f}\left(j\vec{k}\right) := \boldsymbol{\mathcal{F}}^{(3)}_{\vec{r}\to\vec{k}}\{f\} = \iiint_{\mathbb{R}^3} f(\vec{r})\, e^{-j\vec{k}\cdot\vec{r}}\, d^3r \,. \quad (8)$$

Note that a "frown"-symbol has been used instead of a "hat"-symbol on top of the symbol for the transformed function to distinguish it from the ordinary time-frequency Fourier transform.

The inverse transform is given as

$$f(\vec{r}) = \boldsymbol{\mathcal{F}}^{(3)^{-1}}_{\vec{r}\leftarrow\vec{k}}\left\{\widehat{f}\right\} = \frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \widehat{f}\left(j\vec{k}\right) e^{j\vec{k}\cdot\vec{r}}\, d^3k \,. \quad (9)$$

It is not difficult to help learners to understand this procedure: it is a quite simple generalization of the ordinary Fourier transform.

From the defining equations, it follows immediately that

$$\boldsymbol{\mathcal{F}}^{(3)}_{\vec{r}\to\vec{k}}\left\{\operatorname{grad}\psi(t,\vec{r})\right\} = j\vec{k}\,\widehat{\psi}\left(t, j\vec{k}\right), \quad (10)$$

$$\boldsymbol{\mathcal{F}}^{(3)}_{\vec{r}\to\vec{k}}\left\{\operatorname{div}\vec{X}(t,\vec{r})\right\} = j\vec{k}\cdot\widehat{\vec{X}}\left(t, j\vec{k}\right), \quad (11)$$

$$\boldsymbol{\mathcal{F}}^{(3)}_{\vec{r}\to\vec{k}}\left\{\operatorname{curl}\vec{X}(t,\vec{r})\right\} = j\vec{k}\times\widehat{\vec{X}}\left(t, j\vec{k}\right). \quad (12)$$

The benefit of a formulation in reciprocal space is thus obvious: relatively difficult differentiation operations are mapped into relatively simple geometrical operations.

There are impressive successes that have been obtained by application to microscopic models in quantum-electrodynamics (QED) where this formulation was introduced by followers of A. Messiah (see for instance [2]). It is, however, no longer applicable when used to solve Maxwell's equations for a macroscopic model in the interior of material. There, the constitutive relations are neither time-invariant nor space-independent. Therefore, the constitutive relations are still complicated to formulate. These could be better described in the frequency domain.

C. The space-time Fourier transform

While ordinary Fourier transform or Fourier spatial transform are frequently used in separate applications, it appears as if a combination of both was not applied in electrodynamics until now.

Therefore, a fourfold Fourier transform, the space-time Fourier transform, is introduced. It maps functions of the space-time domain onto the range of functions of the frequency and reciprocal space domain:

$$\breve{f}\left(j\omega, j\vec{k}\right) = \iiiint_{\mathbb{R}^4} f(t,\vec{r})\, e^{-j\left(\vec{k}\cdot\vec{r}+\omega t\right)}\, d^3r\, dt = \boldsymbol{\mathcal{F}}^{(4)}_{\substack{t\to\omega\\ \vec{r}\to\vec{k}}}\{f\} \,. \quad (13)$$

Note that a "smile"-symbol on top of the symbol for the transformed function has been used in order to distinguish it from other transformed quantities. The inverse transform is given as

$$f(t,\vec{r}) = \frac{1}{(2\pi)^4}\iiiint_{\mathbb{R}^4} \breve{f}\left(j\omega, j\vec{k}\right) e^{j\left(\vec{k}\cdot\vec{r}+\omega t\right)} d^3k\;\, d\omega = \boldsymbol{\mathcal{F}}^{(4)}_{\substack{t\leftarrow\omega\\ \vec{r}\leftarrow\vec{k}}}\left\{\breve{f}\right\} \,. \quad (14)$$

All the theorems that are known from ordinary Fourier transform are transferred to the Fourier space-time transform.

As a first consequence, it is seen that time derivatives will be mapped to multiplications with the scalar quantity $j\omega$, as it was the case with the ordinary Fourier transform. Gradients of a scalar field will be mapped to a multiplication of the transform of that field with a vector $j\vec{k}$. Next, the divergence of a vector field will be mapped to a scalar product of $j\vec{k}$ and the transform of this field, and the curl of a vector field is mapped to a cross-product of $j\vec{k}$ and the transform of the field. This is exactly as in the case of the Fourier spatial transform. Therefore, the advantages of both transform types are being preserved.

III. MAXWELL'S EQUATIONS IN THE TRANSFORMED SPACE

A. Maxwell's equations

Application of the space-time Fourier transform to Maxwell's equations results in

$$j\vec{k}\cdot\breve{\vec{B}}\left(j\omega, j\vec{k}\right)=0\,, \tag{15}$$

$$j\vec{k}\times\breve{\vec{E}}\left(j\omega, j\vec{k}\right)=-j\omega\,\breve{\vec{B}}\left(j\omega, j\vec{k}\right), \tag{16}$$

$$j\vec{k}\cdot\breve{\vec{D}}\left(j\omega, j\vec{k}\right)=\breve{\rho}\left(j\omega, j\vec{k}\right), \tag{17}$$

$$j\vec{k}\times\breve{\vec{H}}\left(j\omega, j\vec{k}\right)=j\omega\,\breve{\vec{D}}\left(j\omega, j\vec{k}\right)+\breve{\vec{j}}_C\left(j\omega, j\vec{k}\right), \tag{18}$$

where $\breve{\rho}$ is the transform of the space-charge density and where $\breve{\vec{j}}_C$ is the transform of the vector of impressed current density.

The procedure of solving Maxwell’s equations is thus transformed into a procedure of solving a set of linear vector equations that do not contain any differential operation.

B. Some general properties and the continuity equation

Obviously, Maxwell's equations are organized in a way that fields are separated into components parallel and orthogonal to the vector $j\vec{k}$. Following the school of Messiah, the components parallel to $j\vec{k}$ will be called *longitudinal*, while the components orthogonal to it will be called *transverse*. Thus, any vector in the transformed space can be decomposed into a sum of a longitudinal and a transverse vector:

$$\breve{\vec{X}}=\breve{\vec{X}}_{\parallel}+\breve{\vec{X}}_{\perp}\quad;\quad\breve{\vec{X}}_{\parallel}=\left(\vec{k}\cdot\breve{\vec{X}}\right)\vec{k}/k^2\,, \tag{19}$$

apart for the particular case that the absolute value *k* of $j\vec{k}$ is equal to 0. Thus, if the case of position-independent fields is excluded from further considerations, then Maxwell's equations can be reformulated to result in

$$j\vec{k}\cdot\breve{\vec{B}}_{\parallel}=0\,, \tag{20}$$

$$j\vec{k}\times\breve{\vec{E}}_{\perp}=-j\omega\,\breve{\vec{B}}_{\perp}-j\omega\,\breve{\vec{B}}_{\parallel}\,, \tag{21}$$

$$j\vec{k}\cdot\breve{\vec{D}}_{\parallel}=\breve{\rho}\,, \tag{22}$$

$$j\vec{k}\times\breve{\vec{H}}_{\perp}=j\omega\,\breve{\vec{D}}_{\parallel}+\breve{\vec{j}}_{C,\parallel}+j\omega\,\breve{\vec{D}}_{\perp}+\breve{\vec{j}}_{C,\perp}\,. \tag{23}$$

In the above equations, the arguments have been left out for conciseness.

Since the cross-product of $j\vec{k}$ and another vector is always orthogonal to $j\vec{k}$, it must follow from equation (21):

$$0=-j\omega\,\breve{\vec{B}}_{\parallel}\,, \tag{24}$$

which simply re-states what was already known from equation (20). From equation (23), it follows in analogy:

$$0=j\omega\,\breve{\vec{D}}_{\parallel}+\breve{\vec{j}}_{C,\parallel}\,. \tag{25}$$

Multiplying this equation in a scalar product with $j\vec{k}$ and application of equation (22) results in

$$0=j\omega\,\breve{\rho}+j\vec{k}\cdot\breve{\vec{j}}_C\quad\Leftrightarrow\quad 0=\partial\rho/\partial t+\operatorname{div}\vec{j}_C\,, \tag{26}$$

which is the well-known continuity equation.

Obviously Maxwell's equations can be split into two sets of decoupled equations:

$$\breve{B}_{\parallel}=0\,, \tag{27}$$

$$\breve{D}_{\parallel}=\breve{\rho}/(jk)=j_{C,\parallel}/(-j\omega)\,, \tag{28}$$

and

$$j\vec{k}\times\breve{\vec{E}}_{\perp}=-j\omega\,\breve{\vec{B}}_{\perp}\,, \tag{29}$$

$$j\vec{k}\times\breve{\vec{H}}_{\perp}=j\omega\,\breve{\vec{D}}_{\perp}+\breve{\vec{j}}_{C,\perp}\,. \tag{30}$$

These equation sets demonstrate:

1. With given current density and charge density and with given constitutive relations, all field components can be computed.
2. The longitudinal component of the magnetic induction is always zero.
3. The longitudinal component of the electric flux density depends exclusively on the density of free charges.
4. The transverse components of the fields are not independent from one another, apart from the case where $j\omega$ is the zero-operator. Only two scalar quantities – say one component of the transverse magnetic induction and one component of the transverse electric field – will determine the rest of the components together with the two given components of the current density.

It would be difficult to proof this last statement by means of the conventional methods.

While the decomposition of the fields into longitudinal and transverse components deliver some very useful means to reduce the mathematical complexity of electrodynamics, this method suffers from one severe problem: At least until that moment it is not clear to students what is the physical meaning of the vector $j\vec{k}$.

Therefore, another method is sought for that avoids this particular decomposition. This method will be prepared in the next subsection.

C. Electrostatics and magnetostatics

In the last section, it was stated that electric and magnetic field would be decoupled, if

$$j\omega\ \breve{\vec{B}} \equiv 0\,, \tag{31}$$

$$j\omega\ \breve{\vec{D}} \equiv 0\,, \tag{32}$$

i.e., if in the time domain fields would be time-invariant, or

$$\breve{\vec{B}}\left(j\omega, j\vec{k}\right) \equiv 2\pi \widehat{\vec{B}}_{st}\left(j\vec{k}\right)\delta\left(\omega\right), \tag{33}$$

$$\breve{\vec{D}}\left(j\omega, j\vec{k}\right) \equiv 2\pi \widehat{\vec{D}}_{st}\left(j\vec{k}\right)\delta\left(\omega\right). \tag{34}$$

Notice that from equations (31) and (32) it does not follow that the induction and the electric flux density in transformed space vanish everywhere! At frequency 0, they might be different from zero.

This case will be assumed for this subsection. Then, it follows from equations (15) - (18) :

$$j\vec{k}\cdot\breve{\vec{B}} = 0\,, \tag{35}$$

$$j\vec{k}\times\breve{\vec{E}} = -j\omega\ 2\pi\widehat{\vec{B}}_{st}\delta\left(\omega\right), \tag{36}$$

$$j\vec{k}\cdot\breve{\vec{D}} = \breve{\rho}\,, \tag{37}$$

$$j\vec{k}\times\breve{\vec{H}} = j\omega\ 2\pi\widehat{\vec{D}}_{st}\left(j\vec{k}\right)\delta\left(\omega\right) + \breve{\vec{j}}_C\,. \tag{38}$$

Would the transformed current density be different from zero apart from $\omega = 0$, the magnetic field would also be different from zero apart from $\omega = 0$, which contradicts the assumption that $j\omega\breve{\vec{B}}$ be zero. An analogous argument is valid for the charge density. Inverse ordinary Fourier transform thus yields

$$j\vec{k}\times\widehat{\vec{E}} = 0\,, \tag{39}$$

$$j\vec{k}\cdot\widehat{\vec{D}} = \widehat{\rho}_{st}\left(j\vec{k}\right), \tag{40}$$

$$j\vec{k}\cdot\widehat{\vec{B}} = 0\,, \tag{41}$$

$$j\vec{k}\times\widehat{\vec{H}} = \widehat{\vec{j}}_{C,st}\left(j\vec{k}\right). \tag{42}$$

The first two equations describe completely the case of electrostatics, the last two the case of magnetostatics.

From equation (39), it follows that the electric field in reciprocal space must be parallel to the vector $j\vec{k}$. Therefore, it must be possible to express it as follows:

$$\widehat{\vec{E}} = -j\vec{k}\ \widehat{\phi}_{st}\left(j\vec{k}\right), \tag{43}$$

where $\widehat{\phi}_{st}$ is completely determined by the above equation (and where the minus-sign is due to conventions). Inverse Fourier spatial transform immediately shows that then

$$\vec{E}\left(\vec{r}\right) = -\operatorname{grad}\phi_{st}\left(\vec{r}\right). \tag{44}$$

ϕ_{st} is obviously the well-known electrostatic potential function.

In homogeneous, isotropic material, and under static conditions

$$\widehat{\vec{D}} = \varepsilon_{st}\ \widehat{\vec{E}} \tag{45}$$

holds and thus

$$\widehat{\phi}_{st}\left(j\vec{k}\right) = \widehat{\rho}_{st}\left(j\vec{k}\right)\Big/\left(k^2\varepsilon_{st}\right), \tag{46}$$

$$\widehat{\vec{E}} = -j\vec{k}\ \widehat{\rho}_{st}\left(j\vec{k}\right)\Big/\left(k^2\varepsilon_{st}\right). \tag{47}$$

Inverse Fourier spatial transform could be derived on a purely mathematical basis, which is quite laborious. Another solution is simpler.

For a point charge Q in vacuum, it is known that

$$\phi_{st} = Q\big/\left(4\pi\varepsilon_0 r\right), \tag{48}$$

$$\vec{E} = Q\vec{r}\big/\left(4\pi\varepsilon_0 r^3\right), \tag{49}$$

$$\rho = Q\ \delta\left(\vec{r}\right). \tag{50}$$

From that, it follows with $\varepsilon_{st} = \varepsilon_0$ that $\widehat{\rho} = Q$ and comparison with equations (48) and (49) yields

$$\mathcal{F}^{(3)}_{\vec{r}\to\vec{k}}\left\{1/4\pi r\right\} = 1\big/k^2\ , \tag{51}$$

$$\mathcal{F}^{(3)}_{\vec{r}\to\vec{k}}\left\{\vec{r}\big/\left(4\pi r^3\right)\right\} = -\,j\vec{k}\big/k^2\ . \tag{52}$$

Using the convolution theorem of the Fourier transform, it can be concluded from equations (46), (47)

$$\phi_{st}\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_{st}}\iiint_{\mathbb{R}^3}\frac{\rho_{st}\left(\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|}\,d^3r'\,, \tag{53}$$

$$\vec{E}\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_{st}}\iiint_{\mathbb{R}^3}\frac{\rho_{st}\left(\vec{r}\,'\right)\left(\vec{r}-\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|^3}\,d^3r'\,. \tag{54}$$

The electrostatic fields can be derived in a similar way for inhomogeneous, non-isotropic, linear material as well.

The magnetostatic equations are solved in a similar way. After application of a cross-product with $j\vec{k}$ to equation (42), it follows

$$j\vec{k}\left(j\vec{k}\cdot\overset{\frown}{\vec{H}}\right)+k^2\overset{\frown}{\vec{H}}=j\vec{k}\times\overset{\frown}{\vec{j}}_{C,st}\,. \tag{55}$$

In homogeneous, isotropic media, the first sum term in this equation vanishes by virtue of equation (41). Thus, it follows

$$\overset{\frown}{\vec{H}}=\left(j\vec{k}\times\overset{\frown}{\vec{j}}_{C,st}\big/k^2\right). \tag{56}$$

The inverse Fourier spatial transform of the numerator in this equation is $\operatorname{curl}\vec{j}_{C,st}$. The inverse transform of $1/k^2$ was found in equation (51). Application of the convolution theorem of the Fourier transform thus yields

$$\overset{\frown}{\vec{H}}(\vec{r})=\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\frac{\operatorname{curl}\overset{\frown}{\vec{j}}_{C,st}(\vec{r}')}{|\vec{r}'-\vec{r}|}\,d^3r'\,. \tag{57}$$

By substitution of integration variables, another formulation can be derived that is more familiar to experts:

$$\begin{aligned}\overset{\frown}{\vec{H}}(\vec{r})&=\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\frac{\left[\operatorname{curl}_{\vec{X}}\overset{\frown}{\vec{j}}_{C,st}(\vec{X})\right]_{\vec{X}=\vec{r}''+\vec{r}}}{|\vec{r}''|}\,d^3r''\\&=\frac{1}{4\pi}\operatorname{curl}_{\vec{r}}\iiint_{\mathbb{R}^3}\frac{\overset{\frown}{\vec{j}}_{C,st}(\vec{r}''+\vec{r})}{|\vec{r}''|}\,d^3r''\\&=\frac{1}{4\pi}\operatorname{curl}\iiint_{\mathbb{R}^3}\frac{\overset{\frown}{\vec{j}}_{C,st}(\vec{r}')}{|\vec{r}'-\vec{r}|}\,d^3r'\quad.\end{aligned} \tag{58}$$

D. *Potential functions*

The restriction of time-invariant fields is now dropped. In contrast to the decomposition method discussed earlier, a solution of Maxwell's equations is now sought that is independent from a particular coordinate system.

The law of the continuity of magnetic flux in transformed space,

$$j\vec{k}\cdot\breve{\vec{B}}=0, \tag{59}$$

suggests to substitute the transformed magnetic induction by a vector that is *constructed* in a way that it is always orthogonal to $j\vec{k}$:

$$\breve{\vec{B}}=j\vec{k}\times\breve{\vec{A}}. \tag{60}$$

By inverse space-time Fourier transform, it follows

$$\vec{B}=\operatorname{curl}\vec{A}, \tag{61}$$

which reveals the vector $\vec{A}$ as the well-known vector-potential. Its introduction is thus a quite natural development in transformed space.

There, it is immediately understandable that $\breve{\vec{A}}$ is not yet completely determined by the defining equation, since a vector that only has to be orthogonal to a second vector could still rotate in a plane, the normal vector of which is parallel to this second vector. Algebraically, the following relation could express this fact:

$$\breve{\vec{B}}=j\vec{k}\times\breve{\vec{A}}=j\vec{k}\times\left(\breve{\vec{A}}+j\vec{k}\,\breve{U}\right), \tag{62}$$

where $\breve{U}$ is any transformable scalar function in the transformed space. It is thus possible to freely elect the longitudinal component of the transformed vector-potential.

If the vector potential substitutes the magnetic induction, the law of the continuity of magnetic flux is always met automatically. Inserting this substitution into the transformed version of Faraday's law yields

$$j\vec{k}\times\left(\breve{\vec{E}}+j\omega\,\breve{\vec{A}}\right)=0, \tag{63}$$

which means that the term in brackets must be a longitudinal vector. Thus, it must be possible to express this term as

$$\breve{\vec{E}}+j\omega\,\breve{\vec{A}}=-j\vec{k}\,\breve{\phi}, \tag{64}$$

where $\breve{\phi}$ is a scalar function in the transformed space that is defined by the above equation. Because of

$$\breve{\vec{E}}=-j\vec{k}\,\breve{\phi}-j\omega\,\breve{\vec{A}}=-j\vec{k}\left\{\breve{\phi}-j\omega\,\breve{U}\right\}-j\omega\left\{\breve{\vec{A}}+j\vec{k}\,\breve{U}\right\}, \tag{65}$$

an adaptation in the vector potential forces a simultaneous adaptation in $\breve{\phi}$:

$$\breve{\vec{A}}\to\breve{\vec{A}}+j\vec{k}\,\breve{U}\quad;\quad\breve{\phi}\to\breve{\phi}-j\omega\,\breve{U}\,. \tag{66}$$

In ordinary space-time, such a simultaneous adaptation is given by

$$\vec{A}\to\vec{A}+\operatorname{grad}U\quad;\quad\phi\to\phi-\partial\breve{U}/\partial t\,. \tag{67}$$

This formulation is known as gauge transform. Note, that in the transformed space, the gauge transform evolves naturally from vector-algebra, while in ordinary space-time it is not so easy to derive.

Equation (65) shows that in case of a time-invariant vector-potential, $\breve{\phi}$ must be identical to the transformed electrostatic scalar potential, from which it must be concluded that ϕ is the well-known scalar potential.

Equation (65) shows furthermore, how to substitute the electric field by the vector-potential and the scalar potential. By doing so, Faraday's law is met automatically.

It remains to solve Gauss's law and Ampère's generalized law (eqs. (17) and (18)). If it is assumed that the equations are to be solved for homogeneous, isotropic material, they can be formulated as

$$j\vec{k}\cdot\hat{\varepsilon}\breve{\vec{E}} = \breve{\rho}\,, \tag{68}$$

$$j\vec{k}\times\breve{\vec{B}} = j\omega\hat{\varepsilon}\hat{\mu}\breve{\vec{E}} + \hat{\mu}\breve{\vec{j}}_C\,, \tag{69}$$

or after substitution of electric field and magnetic induction by the potential functions:

$$k^2\breve{\phi} - j\vec{k}\cdot j\omega\,\breve{\vec{A}} = \breve{\rho}/\hat{\varepsilon}\,, \tag{70}$$

$$j\vec{k}\times\left(j\vec{k}\times\breve{\vec{A}}\right) - \omega^2\hat{\varepsilon}\hat{\mu}\breve{\vec{A}} + j\omega\hat{\varepsilon}\hat{\mu}\,j\vec{k}\,\breve{\phi} = \hat{\mu}\breve{\vec{j}}_C\,. \tag{71}$$

The last equation can be simplified by expansion of the triple vector product. Therefore, the equations to be solved are:

$$k^2\breve{\phi} - j\omega\;j\vec{k}\cdot\breve{\vec{A}} = \breve{\rho}/\hat{\varepsilon}\,, \tag{72}$$

$$j\vec{k}\,\left\{j\vec{k}\cdot\breve{\vec{A}} + j\omega\,\varepsilon\mu\,\breve{\phi}\right\} + \left\{k^2 - \omega^2\,\varepsilon\mu\right\}\breve{\vec{A}} = \mu\breve{\vec{j}}_C\,. \tag{73}$$

IV. Solution in the Lorentz gauge

It is now the point where the freedom in the choice of the longitudinal component of the vector-potential can be used advantageously in order to decouple the remaining equations.

From a physicist's point of view, it is advantageous to find a solution that is independent from any coordinate system and that applies also for any coordinate system in transformed space.

Inspection of equation (73) shows that this could be done with the following choice that is known as the Lorentz gauge

$$j\vec{k}\cdot\breve{\vec{A}} + j\omega\,\varepsilon\mu\,\breve{\phi} = 0\,, \tag{74}$$

since then

$$\left(k^2 - \omega^2\hat{\varepsilon}\hat{\mu}\right)\breve{\vec{A}} = \hat{\mu}\breve{\vec{j}}_C\,, \tag{75}$$

which by use of Gauss's law (eq. (72)) and the continuity equation (eq. (26)) leads to

$$\left(k^2 - \omega^2\hat{\varepsilon}\hat{\mu}\right)\breve{\phi} = \breve{\rho}/\hat{\varepsilon}\,. \tag{76}$$

Note that equation (76) and the components of equation (75) have the same structure. They can thus be solved using the same algorithm. This will be demonstrated using equation (76). However, it must be hold in mind that the Lorentz gauge (equation (74)) links the components of the vector potential and the scalar potential and that the continuity equation links the density of free space charges and the impressed-current-density.

Using the abbreviation

$$c = 1\big/\sqrt{\hat{\varepsilon}\hat{\mu}}\,, \tag{77}$$

equation (76) is re-formulated as

$$\left\{k - \omega/c\right\}\left\{k + \omega/c\right\}\breve{\phi} = \breve{\rho}/\hat{\varepsilon}\,. \tag{78}$$

Thus

$$\breve{\phi} = \frac{c^2\,\breve{\rho}/\hat{\varepsilon}}{\left(ck+\omega\right)\left(ck-\omega\right)} = \frac{c}{2k}\frac{\breve{\rho}/\hat{\varepsilon}}{\omega+ck} - \frac{c}{2k}\frac{\breve{\rho}/\hat{\varepsilon}}{\omega-ck}\,. \tag{79}$$

The components of the potential in ordinary space-time are found by inverse space-time Fourier transform. Depending on the material, the inverse transform can be more or less complicated, since c might depend on frequency.

A not so complex case is given in vacuum, where permeability and permittivity do not depend on frequency, and where these quantities are both real-valued. In this case, inverse ordinary Fourier transform is not too difficult, since by application of the shifting theorem and the integration theorem it follows:

$$\begin{aligned}\widehat{\phi} = &-\frac{jc}{2k\varepsilon_0}e^{jckt}\left\{\int\limits_{-\infty}^{t}\widehat{\rho}\left(t,j\vec{k}\right)e^{-jck\tau}\;d\tau + \widehat{a}\left(\vec{k}\right)\right\}\\ &+\frac{jc}{2k\varepsilon_0}e^{-jckt}\left\{\int\limits_{-\infty}^{t}\widehat{\rho}\left(t,j\vec{k}\right)e^{jck\tau}\;d\tau + \widehat{b}\left(\vec{k}\right)\right\},\end{aligned} \tag{80}$$

where $\widehat{a}$ and $\widehat{b}$ are integration "constants" that can still depend on $\vec{k}$.

Note, however, that the potential in ordinary time and space is a real-valued field. Thus, the following relation must hold:

$$\widehat{\phi}^*\left(t,j\vec{k}\right) = \widehat{\phi}\left(t,-j\vec{k}\right), \tag{81}$$

which leads to

$$\begin{aligned}\widehat{\phi} = &-\frac{jc}{2k\varepsilon_0}e^{jckt}\left\{\int\limits_{-\infty}^{t}\widehat{\rho}\left(t,j\vec{k}\right)e^{-jck\tau}\;d\tau + \widehat{a}\left(\vec{k}\right)\right\}\\ &+\frac{jc}{2k\varepsilon_0}e^{-jckt}\left\{\int\limits_{-\infty}^{t}\widehat{\rho}\left(t,j\vec{k}\right)e^{jck\tau}\;d\tau + \widehat{a}^*\left(-\vec{k}\right)\right\}.\end{aligned} \tag{82}$$

In ordinary space and time, the scalar potential is found by inverse Fourier spatial transform as:

$$\phi(t,\vec{r})=$$
$$\frac{-1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\frac{jc}{2k\varepsilon_0}e^{jckt}\left\{\int_{-\infty}^{t}\hat{\rho}\left(t,j\vec{k}\right)e^{-jck\tau}d\tau+\hat{a}\left(\vec{k}\right)\right\}e^{j\vec{k}\cdot\vec{r}}d^3k$$
$$+\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\frac{jc}{2k\varepsilon_0}e^{-jckt}\left\{\int_{-\infty}^{t}\hat{\rho}\left(t,j\vec{k}\right)e^{jck\tau}d\tau+\hat{a}^*\left(-\vec{k}\right)\right\}e^{j\vec{k}\cdot\vec{r}}d^3k\ .\quad(83)$$

In a similar way, it follows for the components of the vector-potential

$$A_i(t,\vec{r})=$$
$$\frac{-1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\frac{jc\mu_0}{2k}e^{jckt}\left\{\int_{-\infty}^{t}\hat{j}_i\left(t,j\vec{k}\right)e^{-jck\tau}d\tau+\hat{a}_i\left(\vec{k}\right)\right\}e^{j\vec{k}\cdot\vec{r}}d^3k$$
$$+\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\frac{jc\mu_0}{2k}e^{-jckt}\left\{\int_{-\infty}^{t}\hat{j}_i\left(t,j\vec{k}\right)e^{jck\tau}d\tau+\hat{a}_i^*\left(-\vec{k}\right)\right\}e^{j\vec{k}\cdot\vec{r}}d^3k\ .\quad(84)$$

The evaluation of the above integrals can be a tedious task. However, a similar problem occurs when Maxwell's equations are solved following a traditional method.

Therefore, only a very special case is considered. This is the case of vacuum, vanishing charge density, and vanishing current density. In addition it will be assumed that

$$\hat{a}_1\left(\vec{k}\right)=K\ \delta\left(\vec{k}+k_0\vec{e}_3\right)\quad;\quad\hat{a}_2\left(\vec{k}\right)=\hat{a}_3\left(\vec{k}\right)=0\ .\qquad(85)$$

Then

$$j\vec{k}\cdot\breve{\vec{A}}=0\ ,\qquad(86)$$

from which it follows that the scalar potential vanishes. Furthermore,

$$A_1(t,\vec{r})=\frac{1}{(2\pi)^3}\frac{c|K|}{|k_0|}\sin\left(c|k_0|t-k_0r_3+\measuredangle K\right);$$
$$A_2(t,\vec{r})=0\quad;\quad A_3(t,\vec{r})=0\qquad,\qquad(87)$$

which is a sinusoidal wave traveling into $\vec{e}_3$-direction, which is the direction of $\vec{k}$ for that particular case. This explains, why $\vec{k}$ is also called propagation-vector.

Moreover, since by virtue of the definition theorem of Dirac's delta-operator, any integration-constant $\hat{\vec{a}}$ can be constructed as a superposition of vectors similar to that of equation (85), the general solution of the case without free charges and currents must be a superposition of sinusoidal waves that propagate in any direction of space.

Magnetic induction and electric field are found by virtue of equations (61) and (65), which in the above special case yields

$$B_1(t,\vec{r})=0\quad;\quad B_3(t,\vec{r})=0;$$
$$B_2(t,\vec{r})=-\frac{c|K|}{(2\pi)^3}\frac{k_0}{|k_0|}\cos\left(c|k_0|t-k_0r_3+\measuredangle K\right),\qquad(88)$$
$$E_1(t,\vec{r})=-\frac{c^2|K|}{(2\pi)^3}\cos\left(c|k_0|t-k_0r_3+\measuredangle K\right);$$
$$E_2(t,\vec{r})=0\quad;\quad E_3(t,\vec{r})=0.\qquad(89)$$

Note that in contradistinction to familiar derivations no scientific guesses have been used. Rather a clear statement has been found: In vacuum where there are no charges and current densities, and under the assumption of no boundary conditions, the solution of Maxwell's equations is any *superposition* of plane sinusoidal TEM waves.

V. Conclusions

It was shown that the application of the space-time Fourier transform simplifies mathematics for solving electrodynamical problems considerably. Instead of solving systems of partial vector-valued differential equations, only algebraic vector equations have to be solved. This facilitates the search for independent quantities and thus allows for a more direct insight into physical problems.

A particular advantage of the described procedure is that it yields a relatively complete solution even for more complicated situations as for instance in the theory of simple antennas.

Lecturers will benefit from the novel procedure by being able to concentrate on the explanation of the physical background. Students will benefit by a clear and straightforward mathematical solution of Maxwell's equations that does not make use of scientific guesses.

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