Excess Noise In The Output Of Linear Fluctuating Systems

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Abstract. The impulse-response of many linear systems in biophysics, medicine, electronics, economics and other fields fluctuates randomly; i.e.; it depends not only upon the time-*difference* from the excitation of the impulse; but it depends also randomly upon the absolute point in time the impulse was applied to the system. These systems might be described, therefore, as linear time-variant (LTV) systems. Application of the theory of LTV-systems shows the power-spectrum of the output of such a system, driven by input-noise, contains additional terms as compared to linear time-invariant systems. In this paper a very simple LTV-system is considered that contains four different terms at its output; one term is seen to be proportional to the input-power-spectrum; a second term is proportional to the power spectrum of the system's fluctuations; a third term describes additional equilibrium output-noise. While these three terms are expected to appear in the output-spectrum, the fourth term is less known: It is due to the interactions between the input-noise and the system's fluctuations. It will be shown that this excess noise might contain power at frequencies that did not appear in the input-spectrum, nor in the spectrum of the fluctuations. Its occurrence might potentially explain so-far inexplicable spectra following a $1/f^{\alpha}$-law, and raises, furthermore, numerous new questions.

INTRODUCTION

Linear, noise-generating systems are known from many fields of research as for instance biology, physics, medicine, electronics and economics. The system properties of many of them may fluctuate randomly with time. Examples of such systems are (under certain constraints) ion-channels in biological cells or resistors in semi-conducting material, or flow of goods between two regions with differing prices for those goods.

In this paper, a simple model is considered that takes into consideration possible fluctuations of the transfer behavior. In contrast to the usually assumed time-invariance of the transfer function, a linear time-variance of the transfer function is considered. From a mathematical point of view, time-variant parameters of the describing equations are allowed rather than constant parameters. There exists a well-proven theory of linear time-varying systems, see for example [1, 2], which will be applied to describe the model.

METHODS

The methods applied in this paper are demonstrated with the model depicted in Fig. 1. It has a single, scalar input-signal proportionally linked to the output-signal, where the factor of proportionality is fluctuating with time. An additional internal source of noise exists that describes equilibrium noise (i.e., noise when the system is not driven by an input-signal). In this model, $s(t)$ is the input to the system and $g(t)$ its output.

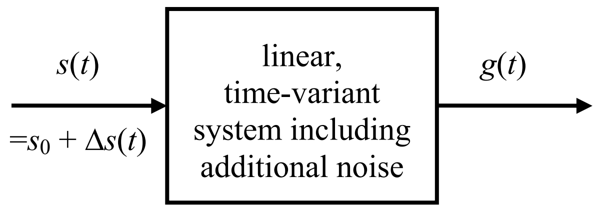

FIGURE 1. Simple linear, time-variant system with fluctuating input

The input is described as a constant term s_0 with a randomly varying term $\Delta s(t)$ superimposed upon it.

$$s(t) = s_0 + \Delta s(t) \quad . \tag{1}$$

The impulse-response of the system under idealized conditions and with respect to an impulse that is excited at point ξ in time is assumed to be:

$$h(t,\xi) = \{h_0 + \Delta h(t)\}\, \delta(t-\xi) \tag{2}$$

The output of the signal is consequently assumed as

$$g(t) = \int_{-\infty}^{\infty} h(t,\xi)\, s(\xi)\, d\xi = \{h_0 + \Delta h(t)\}\{s_0 + \Delta s(t)\} + \Delta g(t) \quad , \tag{3}$$

where $\Delta g(t)$ takes into account fluctuations of the output that are also present, when the input is identically zero (equilibrium fluctuations).

As an example, consider an ohmic resistor with fluctuating resistance $r_0 + \Delta r(t)$ that is driven by a current with average value i_0, and that is subject to the shot effect and thus fluctuates by a value $\Delta i(t)$. This resistance will produce an additional thermal noise voltage $\Delta u(t)$. Thus, the complete output voltage of the resistor is

$$u(t) = \{r_0 + \Delta r(t)\}\{i_0 + \Delta i(t)\} + \Delta u(t) \quad . \tag{4}$$

The aim of this paper is to compare the power spectrum of the input $s(t)$ with that of the output $g(t)$, or in the case of the resistor, to compare the power spectrum of $i(t)$ with that of the output $u(t)$.

In the case of the above example, it is also of interest to compare the spectrum of the power dissipated in the resistor, which is the spectrum evolving from the cross-correlation between input and output. Therefore, the spectra evolving from auto-correlation functions as well as that evolving from the cross-correlation function of a system as described by equation (3) are analyzed in the following.

In order to be able to compute correlation functions and from them spectra, further assumptions on the statistical properties of the processes $\Delta s(t)$, $\Delta g(t)$ and $\Delta h(t)$ are necessary. It will be assumed that all three processes have zero-mean and that they are wide-sense stationary (WSS) and mutually uncorrelated.

Under these conditions, it will then be easy to compute statistical average values and correlation functions:

$$\langle g(t)\rangle = h_0\ s_0 \quad ; \tag{5}$$

$$\mathrm{R}_s(\tau) = s_0^2 + \mathrm{R}_{\Delta s}(\tau) \quad ; \tag{6}$$

$$\mathrm{R}_g(\tau) = h_0^2\left\{s_0^2 + \mathrm{R}_{\Delta s}(\tau)\right\} + \mathrm{R}_{\Delta g}(\tau) + s_0^2\mathrm{R}_{\Delta h}(\tau) + \mathrm{R}_{\Delta h}(\tau)\,\mathrm{R}_{\Delta s}(\tau) \quad ; \tag{7}$$

$$\mathrm{R}_{g,s}(\tau) = h_0\left\{s_0^2 + \mathrm{R}_{\Delta s}(\tau)\right\} \quad ; \tag{8}$$

where $\mathrm{R}_X(\tau)$ is the auto-correlation function of the process X and where $\mathrm{R}_{X,Y}(\tau)$ is the cross-correlation function of the processes X and Y. A Fourier transform of the correlation-functions then yields the studied spectra.

RESULTS AND DISCUSSION

The Output Spectrum

Applying a Fourier transform to the above correlation functions results in:

$$S_s(\omega) = s_0^2 2\pi\,\delta(\omega) + S_{\Delta s}(\omega) \quad ; \tag{9}$$

$$S_g(\omega) = \underbrace{h_0^2\ S_s(\omega)}_{\text{spectrum transferred from input}} + \underbrace{S_{\Delta g}(\omega)}_{\text{equilibrium noise}} + \underbrace{s_0^2 S_{\Delta h}(\omega)}_{\text{spectrum due to fluctuations of impulse-response}} + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\Delta s}(\eta)\,S_{\Delta h}(\omega-\eta)\,d\eta}_{\textbf{excess spectrum}} \quad , \tag{10}$$

$$S_{g,s}(\omega) = h_0 S_s(\omega) \quad . \tag{11}$$

While the first term in the output spectrum S_g is not a surprise at all, and while the next two terms might also be expected, the last term seams to be less well known (Kish has already treated a special case [3], where this term appeared in a different notation that did not lead to a convolution integral).

It is emphasized that the excess-noise spectrum is *not a product of two spectra* as it is often assumed in the literature when dealing with fluctuating systems.

It is pointed out in addition that for the output power-spectrum

$$S_g(\omega) \neq h_0^2\ S_s(\omega) \quad ; \quad S_g(\omega) \neq h_0^2\ S_s(\omega) + S_{\Delta g}(\omega) \quad ; \tag{12}$$

as opposed to assumptions that are found frequently in the literature, although the system is supposed to fluctuate!

As an example, consider an ohmic resistor with fluctuating resistance $r_0 + \Delta r(t)$ that is driven by a current with average value $i_0 + \Delta i(t)$, and which generates a thermal noise voltage $\Delta u(t)$. Then the voltage spectrum is *not* proportional to the current power spectrum. This proportionality might potentially apply as an approach, but it is certainly not the whole truth. For that same example,

$$S_{u,i}(\omega) = r_0 S_i(\omega) \quad ; \tag{13}$$

indicates that the (true) power spectrum follows another rule other than the power-spectrum found from the voltage.

The most important message from equation (10), however, is the fact that there is an excess noise term that might change the spectrum in a very interesting way.

Spectrum-Spreading By An LTV-System

In order to get a first impression of how the output spectrum might be influenced, some simple examples are shown in Figures 2 to 4.

In Figure 2, the power-spectra of the input and the fluctuations of the system are shown together with the excess (!) power-spectrum of the output.

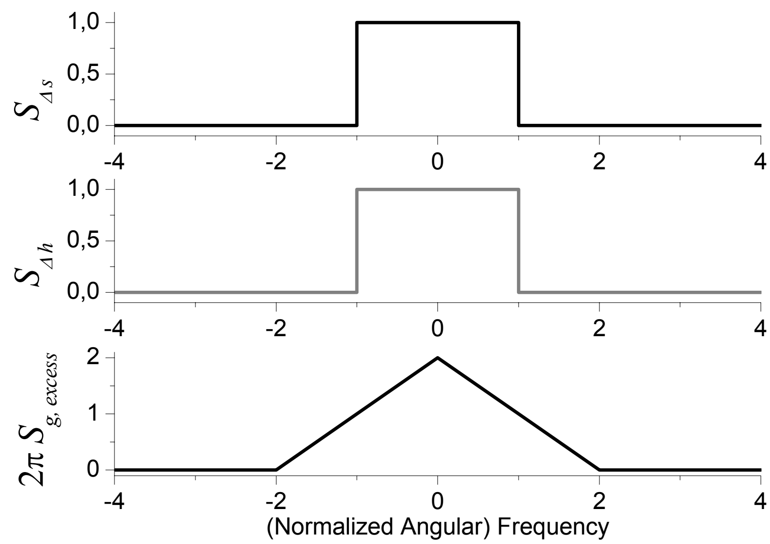

FIGURE 2. Excess spectrum for a simple case

Note that even at frequencies where the power-spectrum of the input as well as that of the fluctuations vanished, the output power-spectrum differs from zero: The output-spectrum is spread as compared to both the input spectrum and the spectrum of the fluctuations. That is definitely a behavior, which is completely unexpected in linear time-*invariant* systems. It is, however, a natural behavior of a linear time-*variant* system.

In a next step, a cascade of two systems each with a behavior identical to that of Fig.2 and with the same input as before is considered. The excess-part of its output power-spectrum is shown in Fig.3.

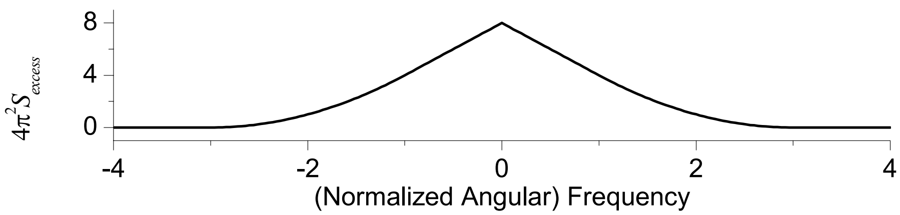

FIGURE 3. Excess spectrum for a simple case

Note that the spectrum is not only spread again, but, additionally, it is smoothed. This behavior develops more and more with an increasing number of stages in the cascade. This is demonstrated in Figure 4, where excess spectra at the output terminals of several stages in a cascade of similar linear fluctuating systems are shown:

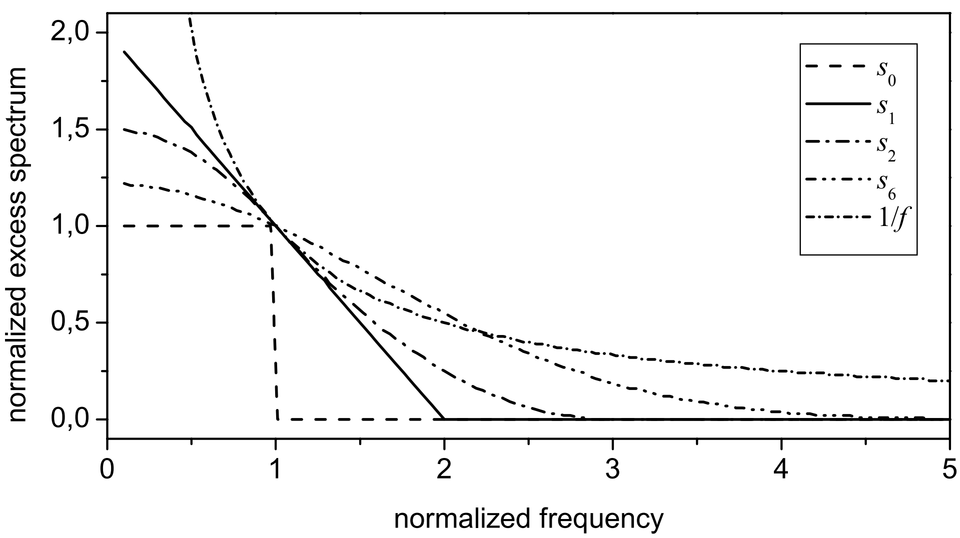

FIGURE 4. Excess spectrum for a simple case

s_0 is the spectrum of the fluctuations in the input of the cascade, s_1 is the excess-spectrum at the output of the first stage, s_2 is the excess spectrum at the output of the second stage and s_6 is the excess spectrum at the output of the sixth stage. For comparison, a $1/f$ curve is shown. All curves are normalized to the value 1 at normalized frequency 1.

Curve s_n vanishes for absolute values of the normalized frequency greater than n+1. All curves s_n are monotonously decreasing. For sufficiently large value of the normalized frequency, $1/f$ is an upper bound for s_n.

The displayed results apply for a very special and not too realistic assumption of the input variations and for the fluctuations of the impulse response. They demonstrate, however, that there might be alterations of the output- spectrum in principle, which have not been taken into consideration so far.

Whether these properties of linear, fluctuating systems might be the key to explain $1/f^{\alpha}$-noise in such systems is an unsolved problem.

OPEN QUESTIONS

There remain numerous other open questions.

Since, traditionally, excess-spectra have not been expected, established practice needs to be re-checked with respect to linear, time-*variant* systems. This is not a triviality, since one of the typical underlying assumptions has been (with vanishing equilibrium fluctuations) S_g be $h_0^2\ S_s$, which has been shown *not* to apply in all cases. Therefore, the following questions are open for discussion again:

1. Can evidence be given for the fluctuations of the parameters of the transfer function in concrete examples? If so,
   - i) What are the statistical properties of these parameters?
   - ii) Can causes be identified for these properties?
2. Based on these findings, will it be possible to design systems with minimized excess noise?
3. With these findings, will it be possible to use that noise as piece of information rather than as disturbance?

Another group of questions is more specific to the basic theory of time-varying systems:

4. How will output spectra be influenced, if there are multi-dimensional input signals and multi-dimensional output signals?
5. How will output spectra be influenced, if the system has memory effects?
6. Will non-linear effects hide or emphasize this behavior?

The most exciting question, however, (at least for this author) is:

7. Might $1/f^{\alpha}$-noise find its explanation as a fundamental property of any distributed (cascaded) time-varying system?

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