1 Introduction

With increasing frequency the length of electromagnetic waves decreases. If the wavelength is equal or smaller than the length of the transmission line, voltage and current can not be assumed as constant over the length of the line. It has to be considered, that voltage and current depend on the $z$ coordinate. In Fig. 1 you see a two-wire line and in Fig. 2 the local voltage curves at three different time points. TEM means transversal electromagnetic, consequently the wave has only transversal field components.

Fig. 1: Equivalent circuit of an ideal TEM transmission line.

Consequences from it:

- voltage and current depend on the $z$ coordinate,
- voltage and current waves move along the line with the time.

The theoretical basics for defining the wavelength on the transmission line and the correlation between current and voltage is the Transmission Line Theory and subject of the next chapter.
2 Theoretical Basics

To cover all voltage and current distributions on transmission lines, we have to solve the Maxwell equations including the boundary conditions. The example below of a two-wire transmission line, however, can be described as an infinitesimal piece of transmission line as shown in the equivalent circuit diagram 3. Consequently, a finitely long transmission line exists from an infinite number of these infinitesimal pieces.

\[ \Delta z \]
\[ u(z, t) \]
\[ \Delta z \]
\[ u(z + \Delta z, t) = u(z, t) + \Delta u(t) \]
\[ i(z + \Delta z, t) = i(z, t) + \Delta i(t) \]
\[ R' \Delta z \]
\[ L' \Delta z \]
\[ G' \Delta z \]
\[ C' \Delta z \]

Fig. 3: Equivalent circuit diagram of an infinitesimal short transmission line.

The relation of current and voltage at the lumped components are:

1. \[ u_R = R' \Delta z i_R, \]  
2. \[ u_L = L' \Delta z \frac{\partial i_L}{\partial t}, \]  
3. \[ i_G = G' \Delta z u_G, \]  
4. \[ i_C = C' \Delta z \frac{\partial u_C}{\partial t}. \]  

Using the mesh rule (KVL=Kirchhoff’s voltage law):

\[ u = R' \Delta z i_R + L' \Delta z \frac{\partial i_L}{\partial t} + u + \Delta u. \]  

Using the node analysis (KCL=Kirchhoff’s current law):

\[ i = G' \Delta z (u + \Delta u) + C' \Delta z \frac{\partial (u + \Delta u)}{\partial t} + i + \Delta i. \]  

Neglecting differences of second order the following equation for the current in valid:

\[ i = G' \Delta z u + C' \Delta z \frac{\partial u}{\partial t} + i + \Delta i. \]
If (5) and (7) are divided by $\Delta z$, in the limit $\Delta z \to 0$; that means the difference equations become differential equations:

$$\frac{\partial u}{\partial z} = - \left( R' + L' \frac{\partial}{\partial t} \right) i,$$

(8)

$$\frac{\partial i}{\partial z} = - \left( G' + C' \frac{\partial}{\partial t} \right) u.$$

(9)

These are the differential equations (DE) of the electrical transmission line forming a system of partial DEs of first order.

In the following we focus on the steady state, that means we deal only with a particular solution of the DEs with sine-shaped excitation:

$$\frac{dI}{dt} = j\omega I,$$

(10)

$$\frac{dU}{dt} = j\omega U.$$

(11)

With the help of these transformation equations (transition to the complex AC calculation) it follows from the DEs (8) and (9):

$$\frac{dU}{dz} = -(R' + j\omega L')I,$$

(12)

$$\frac{dI}{dz} = -(G' + j\omega C')U.$$

(13)

These are the differential equations of electric transmission lines in steady state; it is a linear system of normal DEs of first order with constant coefficients. For the solution of this system of equations we differentiate (12) with respect to $z$ and plug this in (13). So you get the wave equation of this transmission line, the so-called telegraphic equation:

$$\frac{d^2 U}{dz^2} = \gamma^2 U,$$

(14)

with $$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta.$$ (15)

Due to the last relation the following parameters are defined:

$\gamma$: propagation constant

$\alpha$: attenuation constant; for lossless transmission line ($R' = G' = 0$) $\alpha = 0$

is valid

$\beta$: phase constants

$\lambda = \frac{2\pi}{\beta}$: wavelength on the transmission line

$v_{ph} = \frac{\omega}{\beta}$: phase velocity on the transmission line

$v_{gr} = \frac{\partial \omega}{\partial \beta}$: group velocity on the transmission line, on TEM transmission lines it is always valid: $v_{ph} = v_{gr}$. 
Analogous the differential equation for the current holds:
\[
\frac{d^2 I}{dz^2} = \gamma^2 I. \tag{16}
\]

One possible set-up for the solution of the telegraphic equation (14) is:
\[
U(z) = U_{h0}e^{-2\gamma z} + U_{r0}e^{2\gamma z} = U_h(z) + U_r(z). \tag{17}
\]

This set-up can physically be interpreted as a superposition of an incident and a reflected wave. The integration coefficients \(U_{h0}\) and \(U_{r0}\) independent from \(z\) are determined by the boundary conditions of the actual wiring.

(17) plugged in (12) results in the current:
\[
I = -\frac{1}{R' + j\omega L'} \frac{dU}{dz} = \frac{\gamma}{R' + j\omega L'} \left(U_{h0}e^{-2\gamma z} - U_{r0}e^{2\gamma z}\right). \tag{18}
\]

Using the abbreviation:
\[
Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \tag{19}
\]
results in:
\[
I(z) = \frac{1}{Z_0} \left(U_{h0}e^{-2\gamma z} - U_{r0}e^{2\gamma z}\right) = L_{h0}e^{-2\gamma z} - L_{r0}e^{2\gamma z}. \tag{20}
\]

Thus it follows:
\[
\frac{U_h}{I_h} = \frac{U_r}{I_r} = Z_0. \tag{21}
\]

\(Z_0\) is called characteristic impedance. This is the impedance which the wave “faces” when travelling in positive or negative \(z\) direction on an infinitely long transmission line.

The characteristic impedance is assumed to be real in many practical cases. In this experiment, too, we consider this parameter as real.

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**Fig. 4:** Equivalent circuit diagram of a terminated transmission line.
The integration coefficients $U_{h0}$ and $U_{r0}$ are—as mentioned above—determined by the boundary conditions. For example if voltage and current is known at $z=0$ (cf. Fig. 4), hence:

$$U(z=0) = U_a \quad \text{and} \quad I(z=0) = I_a,$$

then the values of voltage and current are determined for each $z$:

$$U(z=0) = U_a = U_{h0} + U_{r0}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (23)$$

$$I(z=0) = I_a = \frac{1}{Z_0}(U_{h0} - U_{r0}), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (24)$$

$$I_a Z_0 = U_{h0} - U_{r0}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (25)$$

$$(23)-(25): \quad U_{r0} = \frac{1}{2}(U_a - I_a Z_0), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (26)$$

$$(23)+(25): \quad U_{h0} = \frac{1}{2}(U_a + I_a Z_0), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (27)$$

$$(17): \quad U = \frac{1}{2}(U_a + I_a Z_0) e^{-\gamma z} + \frac{1}{2}(U_a - I_a Z_0) e^{\gamma z}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (28)$$

$$(18): \quad I = \frac{1}{2Z_0}(U_a + I_a Z_0) e^{-\gamma z} - \frac{1}{2Z_0}(U_a - I_a Z_0) e^{\gamma z}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (29)$$

$$U = U_a \cosh(\gamma z) - Z_0 L_a \sinh(\gamma z), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (30)$$

$$I = I_a \cosh(\gamma z) - \frac{U_a}{Z_0} \sinh(\gamma z). \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (31)$$

And for the impedance at the position $z$, that is $Z(z)$, we get

$$Z(z) = \frac{U(z)}{I(z)} = \frac{U_a \cosh(\gamma z) - Z_0 L_a \sinh(\gamma z)}{I_a \cosh(\gamma z) - \frac{U_a}{Z_0} \sinh(\gamma z)}. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (32)$$

With

$$\frac{U_a}{L_a} = Z_L \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (33)$$

we get

$$Z(z) = Z_0 \frac{Z_L \cosh(\gamma z) - Z_0 \sinh(\gamma z)}{Z_0 \cosh(\gamma z) - Z_L \sinh(\gamma z)} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (34)$$

or

$$Z(z) = Z_0 \frac{Z_L - Z_0 \tanh(\gamma z)}{Z_0 - Z_L \tanh(\gamma z)}. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (35)$$

In case of a loss less transmission line (the losses were already assumed to be that low that the characteristic impedance $Z_0$ was assumed as real) we get

$$\gamma = j\beta, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (36)$$
\[ Z(z) = Z_0 \frac{Z_L - j Z_0 \tan(\beta z)}{Z_0 - j Z_L \tan(\beta z)}. \] (37)

In the following we deal with the problem of finding out the terminating normally complex impedance \( Z_L \) from the voltage characteristic of a TEM transmission line.

Provided that it is a lossfree transmission line, following holds:

\[ R' = G' = 0, \] (38)

and with this: \( \alpha = 0, \quad \gamma = j \beta = j \omega \sqrt{L/C'} \). (39)

For the discussion on the relations of an incident and reflected wave we define the below parameters:

reflection coefficient: \( r = \frac{U_i}{U_h} = -\frac{I_i}{I_h}, \) (40)

\((V)\)SWR: \( s = \left| \frac{U_{\text{max}}}{U_{\text{min}}} \right|, \) (41)

inverse \((V)\)SWR: \( m = \frac{1}{s}. \) (42)

The abbreviation \((V)\)SWR stands for (voltage) standing-wave ratio. Furthermore, the following equations are valid:

\[ |r| = \frac{s - 1}{s + 1}, \] (43)

\[ s = \frac{1 + |r|}{1 - |r|}. \] (44)

Equation (40) plugged into (17) results in:

\[ U = U_{h0} \left( 1 + r_0 e^{2 \gamma z} \right) e^{-\gamma z}, \] (45)

\[ I = \frac{U_{h0}}{Z_0} \left( 1 - r_0 e^{2 \gamma z} \right) e^{-\gamma z}. \] (46)

In case of the boundary condition at \( z=0 \) is given by the load impedance \( Z_L \):

\[ \frac{U(z=0)}{I(z=0)} = \frac{U_i}{I_i} = Z_L \] (47)

we get:

\[ \frac{Z_L}{Z_0} = \frac{1 + r_0}{1 - r_0} \] (48)

\[ \iff \quad r_0 = \frac{Z_L - Z_0}{Z_L + Z_0}. \] (49)

Three frequently appearing special cases are:
open circuit: \( Z_L = \infty \quad r_0 = 1 \quad s = \infty \)

matching condition: \( Z_L = Z_0 \quad r_0 = 0 \quad s = 1 \)

short circuit: \( Z_L = 0 \quad r_0 = -1 \quad s = \infty \)

Matching means that a transmission line having a wave impedance \( Z_0 \) is connected with exactly the same impedance.

Maximum voltages exist at the points where incident and reflected wave are in-phase and their magnitudes add up (constructive interference):

\[
\text{arc} \left( U_{\text{refl}} e^{-j\beta z_{\text{max}}} \right) = \text{arc} \left( U_{\text{inc}} e^{j\beta z_{\text{max}}} \right) = \text{arc} \left( U_{\text{inc}} |r_0| e^{j(\beta z_{\text{max}} + \phi)} \right), \tag{50}
\]

with the unknown reflection coefficient of the DUT:

\( r_0 = |r_0| e^{j\phi}. \tag{51} \)

Hence it results in the condition:

\[-\beta z_{\text{max}} = \beta z_{\text{max}} + \phi + 2n\pi \quad n = 0, \pm 1, \pm 2, \ldots \tag{52} \]

or:

\[ z_{\text{max}} = \frac{\phi}{2\beta} - \frac{n\pi}{\beta} = -\frac{\phi}{2\beta} - n\frac{\lambda}{2} \quad n = 0, \pm 1, \pm 2, \ldots \tag{53} \]

Thus, two neighboured maximum voltages are by \( \frac{\lambda}{2} \) apart.

Analogue calculation for the minimum voltages yields

\[ z_{\text{min}} = -\frac{\phi}{2\beta} - \frac{(2n + 1)\pi}{2\beta} = -\frac{\phi}{2\beta} - (2n + 1)\frac{\lambda}{4} \quad n = 0, \pm 1, \pm 2, \ldots \tag{54} \]

Fig. 5: Upper envelope of the voltage with different load impedances,

a) \( Z_L = 0 \), b) \( Z_L = R_L + jX_L \).

In Fig. 5 the voltage distribution on a transmission line with with a short circuit at the end of the transmission line, curve a), \( Z_L = 0 \), and for a complex load impedance, curve b), is shown.
The voltage distribution means the envelope, the curve of the maximal and minimal voltage $U_E(z)$ (or current $I_E(z)$) which occurs over time. In the figure only the positive voltage is depicted, since the curve of the negative voltage is only the mirror image on the abscissa and do not contain more information. The envelope for both curves can be expressed mathematically as:

$$U_{E+}(z) = \max \left( \hat{U}(z) \cos(\omega t + \phi) \right) = \hat{U}(z),$$

$$U_{E-}(z) = \min \left( \hat{U}(z) \cos(\omega t + \phi) \right) = -\hat{U}(z).$$

The envelope of the current is similar to the curve of the voltage but the course is shifted by $\lambda/4$ with respect to the course of the voltage.

In the case the transmission line was connected to a load not able to convert active power like short-circuit, open-end, capacitor or inductor, the amplitude of the reflected wave has the same value as that of the incident wave. The destructive interference results in points where the voltage is zero, as shown in curve a). In the case of a real or complex load, the reflected amplitude is smaller and no total cancelling can occur, as shown in curve b).

What will be the envelope, if the load impedance is the same as the characteristic impedance of the transmission line?

(Equation 54) results in the following relation between the phase angle of the reflection coefficient $\phi$ at the point $z=0$ and the distance between $z = 0$ and the first minimum voltage $(n=0)$:

$$\phi = -2\beta\Delta z - \pi.$$  

From (42), (43), (44), and (48), we can derive the below relation:

$$\frac{Z_L}{Z_0} = \frac{1 + \frac{1 - m}{1 + m} e^{-j(2\beta\Delta z + \pi)}}{1 - \frac{1 - m}{1 + m} e^{-j(2\beta\Delta z + \pi)}}.$$  

From this it results after some conversions:

$$\frac{Z_L}{Z_0} = \frac{m + j\tan(\beta\Delta z)}{1 + j m \tan(\beta\Delta z)}.$$  

The unknown impedance $Z_L$ can be defined in the case the following three parameters are known: $m$ from (42), the wave impedance of the transmission line $Z_0$, and the relative position

$$\Delta z = z_{\text{min, meas}} - z_{\text{min, SC}}$$

of a minimum (or maximum) voltage with respect to the short circuit.

3 Experiment set-up

On page 14 Fig. 7 you find the schematic test arrangement. The 3 GHz oscillator provides the basic frequency for the measurement. The frequency is amplitude modulated by a 1kHz
oscillator and a PIN diode switch. The isolator and the attenuators protect the components from reflected waves. The probe detects the distribution of the voltage on the measuring transmission line and by means of the detector diode and the bandpass filter the 1 kHz modulated frequency is filtered out, amplified and finally measured.

The modulation is not really necessary, but an alternating voltage of this low frequency is much easier to handle than a direct voltage.

The transmission line used for measurement consists of two parallel plates arranged on the same potential and of a round internal conductor. In Fig. 6 the scheme for measuring transmission line without sockets and the probe in comparision with a coaxial line are shown.

![Fig. 6: a) Coaxial line and b) Transmission line for measuring.](image-url)
4 Questions and Problems on the Lab

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**Problem 1:** Write down the expression for a forward wave and a backward wave separately.

**Problem 2:** Two wires several wavelengths long serve as a connection between a generator and a load. The distance between the wires is small but not constant, varying as a smooth function along the line.
   Can you use the transmission line equation (17) for the analysis of this transmission line?

**Problem 3:** What are the input impedances of lossless transmission lines of lengths \( \frac{\lambda}{4} \) and \( \frac{\lambda}{2} \) in case they are short-circuited or open-ended?

**Problem 4:** What is the effect on the graph of the voltage in Fig. 5 a), in the case the transmission line is not short-circuited but open-ended?

**Problem 5:** How many minima and maxima are detectable if a transmission line of length \( l=15 \text{ cm} \) is short circuited at a frequency of \( f=3 \text{ GHz} \) (the relative permittivity is \( \varepsilon_r=1 \)). Sketch the envelope (voltage characteristic) and mark the minima as well as the maxima. How many minima and maxima are detectable if the transmission line is open ended and \( \varepsilon_r=4 \)?

**Problem 6:** Define the range of values which the reflection coefficient \( r=|r|e^{i\phi} \) can reach with inductive or capacitive load using a Smith chart. Afterwards, determine the shift of the minimum with inductive or capacitive load using (54)?

**Problem 7:** What is more useful for the definition of the wavelength; to measure the minima or the maxima of the field distribution?
5 Measuring Tasks

Make sure that the experimental set-up is built according to Fig. 7. The RF components and the golden SMA connections are damageable and expensive, therefore, please don’t dismantle the connections. On the voltage meter the unit is indicated as $\mu$A. Anyway it is a voltage meter controlled by the amplified and filtered diode signal. Since only the relative values are of interest, please take a.u. (arbitrary unit) as unit. Pay attention to the algebraic sign of the measured distances!

Useful formulas:

- $Z = \frac{m + j \tan(\beta \Delta z)}{1 + jm \tan(\beta \Delta z)} Z_0$
- $Z_0 = 50 \Omega$
- $\Delta z = z_{\text{max}} - z_{\text{SC}}$
- $z_{\text{max}} = \text{Position of the measured maximum}$
- $z_{\text{SC}} = \text{Position of the first measured maximum at the calibration with a short circuit}$
- $m = \frac{|U_{\text{min}}|}{|U_{\text{max}}|}$

Calibration:

Unscrew the probe slightly with the knurled screw. Connect a short circuit to the transmission line; move the measuring slide to the global maximum and revolve the probe till the voltmeter shows full-scale deflection. After this calibration the depth of penetration of the probe must not be changed! Not the position of the first maximum $z_{\text{SC}}$.

Task 1: Calibrate the measurement device at the global maximum.
Determine the locations and the appropriate amplitude of the first two maxima and minima.
Measure now without a new calibration the locations and amplitudes for the case the transmission line is open-ended.
Why do the amplitudes of the maxima differ?

Task 2: Determine as exactly as possible the wavelength on the transmission line using the results from the previous task and from this value the frequency of the RF source. The frequency of the RF source of set-up 1 is 2.999 GHz, that one of set-up 2 3.003 GHz. Calculate the relative error.

Task 3: Connect the 50 $\Omega$ N-connector impedance (this is the value of the characteristic impedance of the transmission line) to the transmission line and determine the standing wave ratio SWR. Calculate the amplitude of the reflection coefficient in dB. What is theoretical value for the standing wave ratio and the reflection coefficient in this case? What are the reasons for the difference between theory and practice?

Task 4: The impedance $Z_2$ (marked with four dots in the brass bulk) consists of a metal film resistor of 100 $\Omega$ which acts only at low frequencies (below the MHz range) as wanted. Determine the complex impedance and from this a simple equivalent circuit consisting of $R$ and $L$ or $C$ for the given frequency. Determine the values of the equivalent circuit.

Task 5: The impedance $Z_1$ (marked with one dot in the brass bulk) consists of a short
open-ended coaxial transmission line, filled with the dielectric media with $\varepsilon_r=2.0$. The characteristic impedance of the coaxial transmission line is 50 $\Omega$ as of the air-filled transmission line.

Determine the complex impedance of the circuit and from this the length of the transmission line.

Hint: $\arccot(x) = \frac{\pi}{2} - \arctan(x)$.

**Task 6:** Calibrate the transmission line using an SMA short circuit and an N-SMA transition.

**Task 7:** Determine the value of the SMD capacitor.

**Task 8:** Determine the values of SWR as well as the magnitude of the reflection factor in dB for the four configurations of SMD resistors (1 $\times$ 50 $\Omega$, 2 $\times$ 100 $\Omega$ in parallel, 3 $\times$ 150 $\Omega$ in parallel and 4 $\times$ 200 $\Omega$ in parallel).

Calculate the theoretical value of the reflection factor in dB for the case the load resistor has 55 $\Omega$.

Why the measured results for the resistor configurations differ?

**Task 9:** Where are the limits of the measuring method and what errors exist? Why the measurement results are not ideal?
Fig. 7: Experimental set-up.