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Must our models on thermal noise be revised?

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ABSTRACT

It will be shown in this paper that our models on thermal noise are subject to strong restrictions that prevent their application at very high frequencies. It appears therefore that there is a need for a modification of our models predicting thermal noise.

INTRODUCTION

Thermal noise is a well-known phenomenon in physics and electronics engineering. It was first reported 1928 by Johnson [1]. Experiments that were performed with different types of resistors and for frequencies between approximately 300 Hz and approximately 2 kHz could be well described by an assumed noise-power spectral density

$$N_{Johnson}(w/2p) = kT \quad , \qquad (1)$$

where k is Boltzmann's constant and T is the temperature in K.

A first explanation was given in the same year by Nyquist [2]. Nyquist's first model predicted exactly Johnson's noise power density, based on the assumption of equipartition of energy.

However, since this model would request infinite over-all-power, Nyquist adapted his model without giving any proof by following Planck's law of black-body radiation:

$$N_{Nyquist}\left(\frac{w}{2p}\right) = \frac{\hbar w}{e^{\hbar w/kT} - 1} \quad . \qquad (2)$$

In 1951, Callen and Welton [3] found a relationship between equilibrium fluctuations in a quantum system and irreversibility. This relationship is often referred to as the "fluctuation-dissipation-theorem". Applied to impedances, it predicted fluctuations of the current and the noise-power spectral density

$$N_{CW}\left(\frac{w}{2p}\right) = \frac{\hbar w}{e^{\hbar w/kT} - 1} + \frac{\hbar w}{2} \quad . \qquad (3)$$

If $\hbar\omega$ is small as compared to kT, then

$$N_{CW}(w/2p) \approx N_{Nyquist}(w/2p) \approx N_{Johnson}(w/2p) = kT \quad . \qquad (4)$$

However, the fluctuation dissipation theorem is difficult to interpret for very high frequencies, since it seemingly predicts increasing power-density for increasing frequencies. There were passionate discussions concerning this behavior that were summarized by Abbott et al [4] or by Reggiani et al [5]. (We are not going to repeat this discussion here).

This suggests that there are some restrictions or even inaccuracies in Callen's and Welton's model (CW-model) that are not yet analyzed. This model applies a first-order perturbation approach to a quantum system that is supposed to act as an "impedance". Inaccuracies could therefore possibly be observed in the modeling of the system or in the describing equations (i.e. in the Hamiltonian of the system) or in the application of the perturbation approach.

Therefore, the Hamiltonian used in the CW-model will be compared in this paper to a more specialized version of a Hamiltonian of an "impedance" in the environment of a "heat reservoir" and with an external field. As a consequence, some restrictions to the applicability of the model will be recognized.

CALLEN'S AND WELTON'S HAMILTONIAN

Callen and Welton used in their original paper on "Irreversibility and Generalized Noise" [3] the following Hamiltonian

$$H^{(CW)} = H_0^{(CW)} + V(t)\,Q(\ldots, q_a, \ldots, \vec{p}_a, \ldots) \quad . \qquad (5)$$

They assumed V to be a simple function of time, Q an operator depending on canonical coordinates and momenta, not on time. From their further explanations concerning the application to thermal noise, it could be derived that $V(t)$ was interpreted as a voltage across an impedance while Q described charge carriers.

It should be mentioned that this particular Hamiltonian was used as a working hypothesis:

no evidence was given for the assumption that this was indeed a Hamiltonian that could describe an impedance approximately, let alone exactly.

Therefore, we will derive in the next two sections a model of an impedance and a Hamiltonian that could be applied to this impedance.

MODEL OF AN "IMPEDANCE"

Figure 1 shows the schematic diagram of an "impedance" that is in contact with a heat reservoir. Two terminal planes allow for proper definition of quantities to be measured, e.g. current or modes of electromagnetic waves.

In order to simplify the system, it is assumed that the essential parts of the system can be modeled as a sequence of very thin slices cut in transverse direction to the z-axis and at a certain position z. Each slice shall again be cut into parallel stripes, the sub-impedances.

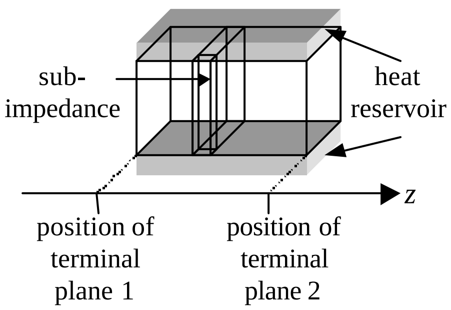

Fig. 1: Model of an impedance that is in contact to a heat reservoir

Since each slice is cut into many stripes, statistical average values can be determined for the current through the front or back areas of the stripes, thus resulting in an expected value of a particle current through the front or back areas or through the side areas of a slice that is in contact to the heat reservoir.

The entire impedance is thus subdivided into a large set of sub-impedances with same properties. These will be analyzed in the following.

THE HAMILTONIAN OF A STRIPE

In this section, the Hamiltonian of a stripe (sub-impedance) will be described. This description is a summary of known results that can be found in reference books on quantum electrodynamics (see e.g. [6]).

On a microscopic scale, and in a classical approach, the Hamiltonian of one sub-impedance including influences of the reservoir would be in a non-relativistic approach

$$H=\sum_a \frac{m_a \dot{\vec{r}}_a^2(t)}{2}+\frac{\varepsilon_0}{2}\int d^3r\left[\vec{E}^2(\vec{r},t)+c^2\vec{B}^2(\vec{r},t)\right] \tag{6}$$

where m_a and $m_a\dot{\vec{r}}_a$ are the masses and the kinetic momenta of the particles. $\vec{E}$ and $\vec{B}$ are the electric field and the magnetic induction, c is the velocity of light in vacuum, ε_0 is the permittivity.

The electromagnetic field is governed by Maxwell's equations and by the Newton-Lorentz equations of the particles. These equations are usually decoupled by introduction of a vector potential $\vec{A}$:

$$\nabla\times\vec{A}(\vec{r},t)=\vec{B}(\vec{r},t) \quad , \tag{7}$$

and of a scalar potential Φ:

$$\nabla\Phi(\vec{r},t)=-\frac{\partial}{\partial t}\vec{A}(\vec{r},t)-\vec{E}(\vec{r},t) \quad . \tag{8}$$

Both potentials are not yet uniquely defined by these equations. Using the Coulomb gauge transformation

$$\nabla\cdot\vec{A}(\vec{r},t)=0 \tag{9}$$

yields another equation to determine the fields. In that case, the potentials must obey the equations

$$\Delta\Phi(\vec{r},t)=-\frac{1}{\varepsilon_0}\rho(\vec{r},t) \quad , \tag{10}$$

$$\frac{\partial^2}{\partial t^2}\vec{A}(\vec{r},t)-c^2\Delta\vec{A}(\vec{r},t)=\frac{1}{\varepsilon_0}\vec{j}(\vec{r},t)-\nabla\frac{\partial}{\partial t}\Phi(\vec{r},t) \tag{11}$$

where

$$\rho(\vec{r},t)=\sum_a q_a\,\delta\left[\vec{r}-\vec{r}_a(t)\right] \tag{12}$$

is the charge density and

$$\vec{j}(\vec{r},t)=\sum_a q_a\,\dot{\vec{r}}_a(t)\,\delta\left[\vec{r}-\vec{r}_a(t)\right] \tag{13}$$

is the (particle) current density. That means: there will be a non-vanishing vector potential for a non-vanishing current density.

For a more lucid analysis of the effects of impressed fields and currents, vector potential and scalar potential will be split into a sum of components of the system (index s) and of external quantities (index e):

$$\vec{A}(\vec{r},t)=\vec{A}_e(\vec{r},t)+\vec{A}_s(\vec{r},t) \quad , \tag{14}$$

$$\Phi(\vec{r},t)=\Phi_e(\vec{r},t)+\Phi_s(\vec{r},t) \quad . \tag{15}$$

The external fields must obey the equations

$$\Delta\Phi_e + \Delta\cdot\vec{A}_e = -\frac{1}{\varepsilon_0}\rho_e \quad , \tag{16}$$

$$\frac{\partial^2}{\partial t^2}\vec{A}_e - c^2\,\Delta\vec{A}_e + c^2\nabla\left(\nabla\cdot\vec{A}_e\right) + \nabla\frac{\partial}{\partial t}\Phi_e = \frac{1}{\varepsilon_0}\vec{j}_e \quad . \tag{17}$$

Again, current density and vector potential are strongly correlated.

Another gauge transformation (not necessarily a Coulomb gauge transformation) can be used to decouple the equations.

The Hamiltonian can then be expressed as

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left[\vec{p}_\alpha - q_\alpha\,\vec{A}(\vec{r}_\alpha,t)\right]^2 + V_{Coul} + \sum_\alpha q_\alpha\,\Phi_e(\vec{r}_\alpha,t)$$
$$+\frac{\varepsilon_0}{2}\int d^3r\left[\left(\frac{\partial}{\partial t}\vec{A}(\vec{r},t)\right)^2 + c^2\left(\nabla\times\vec{A}(\vec{r},t)\right)^2\right] \tag{18}$$

with

$$\vec{p}_\alpha = m_\alpha\,\dot{\vec{r}}_\alpha + q_\alpha\,\vec{A}(\vec{r}_\alpha,t) \tag{19}$$

being the canonically conjugate momenta and

$$V_{Coul} = \frac{1}{8\pi\varepsilon_0}\sum_{\alpha\neq b}\frac{q_\alpha q_b}{\left|\vec{r}_\alpha - \vec{r}_b\right|} + \sum_\alpha\frac{q_\alpha^2}{16\varepsilon_0\pi^3}\int\frac{d^3k}{k^2} \tag{20}$$

being the Coulomb interaction energy between pairs of particles plus the Coulomb self energy.

Carrying on to a quantum-theoretical analysis, the classical total energy is used to gain the hamiltonian operator H, that formally looks like the Hamiltonian given above, apart from spin-effects. It relates the (quantum-theoretical multi-particle) state vector $\left|\psi(t)\right\rangle$ of the system and its time derivative by Schrödinger's equation:

$$j\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle \quad . \tag{21}$$

However, since in classical systems the spin of the particles is not taken into consideration, the Hamiltonian has to be completed:

$$H = \sum_\alpha\frac{1}{2m_\alpha}\left[\vec{p}_\alpha - q_\alpha\,\vec{A}(\vec{r}_\alpha,t)\right]^2 + V_{Coul}$$
$$+\sum_\alpha q_\alpha\,\Phi_e(\vec{r}_\alpha,t) - \sum_\alpha\frac{g_\alpha q_\alpha}{2m_\alpha}\vec{S}_\alpha\cdot\left\{\nabla\times\vec{A}\right\}\Big|_{\vec{r}_\alpha}$$
$$+\frac{\varepsilon_0}{2}\int d^3r\left[\left(\frac{\partial}{\partial t}\vec{A}(\vec{r},t)\right)^2 + c^2\left(\nabla\times\vec{A}(\vec{r},t)\right)^2\right] \tag{22}$$

where g_α is the Landé-factor for the particle and $\vec{S}_\alpha$ the spin-operator associated to the particle with number α.

If the external fields vanish (i.e. if the influence of the heat reservoir and the surrounding sub-impedances vanish), then the Hamiltonian is called undisturbed. It will be named H_0. It then follows

$$H = H_0 + \sum_\alpha q_\alpha\,\Phi_e - \sum_\alpha\frac{g_\alpha q_\alpha}{2m_\alpha}\vec{S}_\alpha\cdot\left\{\nabla\times\vec{A}_e\right\}\Big|_{\vec{r}_\alpha}$$
$$-\sum_\alpha\frac{q_\alpha}{m_\alpha}\left[\left(\vec{p}_\alpha - q_\alpha\,\vec{A}_s\right)\cdot\vec{A}_e - q_\alpha\,\vec{A}_e^2\right]$$
$$+\frac{\varepsilon_0}{2}\int d^3r\left[\left(\dot{\vec{A}}_e\right)^2 + c^2\left(\nabla\times\vec{A}_e\right)^2\right]$$
$$+\varepsilon_0\int d^3r\left[\dot{\vec{A}}_e\cdot\dot{\vec{A}}_s + c^2\left(\nabla\times\vec{A}_e\right)\cdot\left(\nabla\times\vec{A}_s\right)\right] \tag{23}$$

This Hamiltonian is certainly different from that given by Callen and Welton. The perturbation part can not be separated into a product of a time-function and a time-invariant operator.

RESTRICTIONS

In order to simplify the model, we will now introduce some restrictions.

In the appendix of [7] it is shown, that the interaction between spins and magnetic induction can be neglected against the first-order particle-field-interactions for low-energy photons, where the momentum of the photon is small compared with the momentum of the particles. This applies for example for the interaction between bound electrons and microwave photons. The influence of the spin-operator to the Hamiltonian will therefore no longer be considered.

In the same way, the second-order particle-field-interactions will be neglected against the first-order particle-field-interactions, since it will be supposed that radiation intensities are sufficiently low. Furthermore, interactions between external and internal fields will be neglected.

With these restrictions, the Hamiltonian of a sub-impedance can be expressed as

$$H = H_0 + \sum_\alpha q_\alpha\left[\Phi_e(\vec{r}_\alpha,t) - \frac{1}{m_\alpha}\vec{p}_\alpha\,\vec{A}_e(\vec{r}_\alpha,t)\right]$$
$$+\frac{\varepsilon_0}{2}\int d^3r\left[\left(\frac{\partial}{\partial t}\vec{A}_e(\vec{r},t)\right)^2 + c^2\left(\nabla\times\vec{A}_e(\vec{r},t)\right)^2\right]. \tag{24}$$

If electromagnetic wave-propagation in the inner of the impedance could be neglected, the last sum term in equation (24) could also be neglected, leading to

$$H = H_0 + \sum_a q_a \left[\Phi_e(\vec{r}_a, t) - \frac{1}{m_a} \vec{p}_a \cdot \vec{A}_e(\vec{r}_a, t) \right]. \quad (25)$$

For sufficiently small absolute values of the external vector potential this Hamiltonian could again be simplified to result in

$$H = H_0 + \sum_a q_a \Phi_e(\vec{r}_a, t) \quad . \quad (26)$$

Because of its importance for the following argumentation, the assumed restrictions are summarized:

- momentum of photon is small compared with momentum of particles,
- radiation intensities are sufficiently low,
- electromagnetic wave-propagation in the interior of the impedance is negligible,
- external vector potential negligible, i.e. sufficiently small current or frequency.

It is furthermore important to have in mind that the external scalar and vector potential mediate not only power and particle flow in z-direction but also in a direction perpendicular to z. They are therefore also responsible for the heat transport from and to the reservoir and the surrounding sub-impedances. Note thus that even for low frequencies, differences in Φ_e must not be interpreted as voltage across the sub-impedance.

COMPARISON TO THE CW-MODEL

Direct comparison of the simplified Hamiltonian following equation (26) to Callen's and Welton's Hamiltonian shows that the latter can only be a still more simplified version of the system's Hamiltonian.

Indeed, if it is assumed that

$$\Phi_e(\vec{r}, t) = V(t)\, f(z) + \Phi_{eHR}(\vec{r}, t) \quad , \quad (27)$$

where the last sum term takes into account the influence of the heat reservoir, then we can write

$$H = H_0 + V(t) Q(\vec{r}_a) + \sum_a q_a \Phi_{eHR}(\vec{r}_a, t) \quad (28)$$

with

$$Q(\vec{r}_a) = \sum_a q_a\, f(z_a) \quad . \quad (29)$$

Callen and Welton interpreted VQ as a perturbation to the undisturbed Hamiltonian H_0. Application of a perturbation calculation led to a transition probability for bringing the system into a higher state of energy. The latter must then be balanced by heat transport to the heat reservoir. Applied to an ensemble of sub-impedances, this resulted finally in their formula (3).

CONCLUSIONS

If the restrictions are considered that must be met for a sensible application of Callen's and Welton's model, then it is obvious that *we should not expect the CW-formula to apply for radiation dominated processes or to processes where substantial currents flow.*

That could resolve some discrepancies that are described in the literature. Koch et al. [8] for instance seem to have found an experimental verification of Callen's and Welton's model for a resistively shunted Josephson junction and for frequencies up to more than 150 GHz. It seems that this experiment was particle dominated. This is opposed to other experiments that were reported by Gardiner [9] who writes that the increasing part of the spectrum could not be measured by means involving the absorption of photons from a radiation field. Obviously the latter experiments were radiation dominated.

What is still missing is a model that yields the noise power spectral density as well in the particle dominated as in the radiation dominated case. It appears thus that our models on thermal noise must be revised.

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