

A transform method to teach Maxwell-Theory

Michael H.W. Hoffmann

University of Ulm, Department of Microwave Techniques
Ulm, Germany

ABSTRACT: In electronics and information engineering (EIE), teaching Maxwell-theory is always a challenge, since this theory accumulates mathematical and physics problems at the same time. As a result, students will see mainly the mathematical problems of the theory. A deeper insight into the physics behind the theory will be made difficult. Consequently, only few EIE students will show the necessary comprehension of this material. In this paper, a mathematical method based on a fourfold Fourier transform is introduced that makes use of the improved mathematical education that EIE students nowadays receive, particularly in signal processing. It maps partial differential equations completely into simple algebraic equations, and these can be solved easily. As a result of this method, the lecturer is enabled to concentrate more on physics and engineering instead of mathematical difficulties. Students will thus gain a better comprehension of the material.The method will be described in more detail. Some examples will be given to demonstrate its usefulness and to show its didactic benefits.

INTRODUCTION

Lectures on electrodynamics have always been a challenge, not only for students but also for lecturers, since the subject combines several mathematical and physical difficulties at the same time. Thus, it happens frequently that students only see the mathematical problems without getting a deeper understanding of the physics behind them.

Therefore, a procedure is proposed that reduces mathematical problems. The price for this simplification is the introduction of a mathematical tool that normally is only used in a less complex form. This tool is the application of a combination of a Fourier transform from the domain of time-domain functions to the range of frequency-domain functions and of a triple Fourier transform from the domain of ordinary-space-domain functions to the range of reciprocal-space-domain functions. It appears, however, as if this price was not too high, since the application of Fourier transforms is nowadays an everyday job even for undergraduate students in their last year.

Why should this procedure be advantageous?

It is because it transforms a very abstract description using partial differential equations completely into a more tangible geometrical description. It thus stimulates the powers of imagination and opens the mind for interpretation instead of establishing an attitude of mechanically applying mathematical algorithms. This procedure shall be outlined in the following.

THE ORDINARY AND THE SPATIAL TRANSFORM

Every lecturer knows about the extraordinary usefulness of Fourier transforms, e.g. in finding solutions of linear differential equations or in finding an appropriate description of relations that depend on the frequency of an exciting sinusoidal signal.

In principle, this applies also for the formulation of basic relations in electrodynamics. We are used, for example, to write the constitutive relations between the complex phasors of electric field and electric flux density or between magnetic field and magnetic induction of so-called linear material as

$$\vec{D} = \varepsilon \vec{E} \quad \text{and} \quad \vec{B} = \mu \vec{H} .$$

However, every student who has passed a course on signal processing will be able to see that a linear phasor-relationship in reality is a relation between Fourier transforms of time-domain functions.

Once we have accepted this view, we should use a more consistent way to distinguish between time-domain functions and frequency domain functions. Thus let us call the (ordinary) Fourier transform of a time-domain function $f(t)$:

$$\hat{f}(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt .$$

Students see then at once that the above relations between the fields should be reformulated as

$$\hat{\vec{D}} = \hat{\varepsilon}\, \hat{\vec{E}} \quad \text{and} \quad \hat{\vec{B}} = \hat{\mu}\, \hat{\vec{H}}$$

This seems to be a very formal and pedantic viewpoint. But it will turn out that it will help us in finding a new understanding of electrodynamics.
We can interpret equations like these, for instance, in complete analogy to the relation between input and output of a linear

circuit in electronics. The electric flux density in matter could thus be understood as a reaction of the material to the electric field. The permittivity is then to be interpreted as a transfer function between the electric field and the electric flux density. And this is an interpretation that is particularly adapted to the way how engineering students use to learn.

Every sophomore student of electrical and information engineering knows that a transfer function is the Fourier transform of an impulse response which in its turn is mostly the solution of a differential equation.

We can thus be quite sure that students accept that the Fourier transform is an extremely useful means for solving differential equations. And that guides us to the idea to apply such a method to differential equations that depend on other variables than time *t*. Let us consider for instance one of the well-known relations of electrostatics. Here, the electric field can be derived form a scalar potential function $\phi(r_1,r_2,r_3)$:

$$\vec{E}_{static} = -\operatorname{grad}\phi = -\left(\frac{\partial\phi}{\partial r_1},\frac{\partial\phi}{\partial r_3},\frac{\partial\phi}{\partial r_3}\right).$$

I.e., the first component of the electrostatic field is given by the differential equation

$$E_{static,1} = -\frac{\partial\phi}{\partial r_1}.$$

How could we benefit of the theory of Fourier transforms in this case? The answer is quite simple. From a mathematical point of view, the time variable *t* in the ordinary Fourier transform is just any real-valued variable. And the variable of angular frequency, ω, is also simply a real-valued variable.

Thus, if we replace these variables by other real-valued variables, let us say r_1 and k_1, then nothing is changed but the notation. So let us call such a transform of a function $g(r_1)$

$$g^{(1)}(jk_1) = \int_{-\infty}^{\infty} g(r_1)\ \ e^{-jk_1 r_1} dr_1 .$$

Then the Fourier transform of the first component of the electrostatic field with respect to functions from the r_1-range is

$$E^{(1)}_{static,1} = -jk_1\phi^{(1)}.$$

Again, we have replaced a derivative (in the domain of r_1-functions) by a multiplication (in the range of k_1-functions). However, since the coordinates r_2 and r_3 are independent from r_1, it follows for the Fourier transform of the electrostatic field vector:

$$\vec{E}^{(1)}_{static} = -\left(jk_1\phi^{(1)}, \partial\phi^{(1)}/\partial r_2, \partial\phi^{(1)}/\partial r_3\right).$$

I.e., we have eliminated only one of three derivatives. Thus, if we want to eliminate the two other ones, we will have to perform two more Fourier transforms, one with respect to r_2-domain functions and another one with respect to r_3-domain functions. In summary, we have to apply three transforms:

$$\hat{f}(jk_1, jk_2, jk_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r_1,r_2,r_3)e^{-jk_1r_1}dr_1\, e^{-jk_2r_2}dr_2\, e^{-jk_3r_3}dr_3$$

or in a more concise form:

$$\hat{f}\left(j\vec{k}\right) = \int_{\mathbb{R}^3} f(\vec{r})e^{-j\vec{k}\cdot\vec{r}}\ d^3r$$

where we have used vector notation. This threefold Fourier transform of space-domain functions is known as the *Fourier-spatial transform,* the range of transformed functions is called *reciprocal space*. This transform is applied with great success in physics; see for instance [1]. If we apply it to the electrostatic field, then it follows immediately:

$$\hat{\vec{E}}_{static} = -\left(jk_1\phi^{(1)}, jk_2\phi^{(1)}, jk_3\phi^{(1)}\right) = -j\vec{k}\ \phi^{(1)}.$$

We see that the gradient-operator acting on any scalar space-domain function ψ is replaced by a multiplication with the vector $j\vec{k}$:

$$\operatorname{grad}\psi \quad\mapsto\quad j\vec{k}\ \psi .$$

It will turn out that for a fixed angular frequency ω, electromagnetic waves will propagate in parallel to the vector $\vec{k}$. Therefore, we will call it *propagation vector*.

In analogy, it follows that the divergence and the curl of any vector in the space-domain are transformed to

$$\operatorname{div}\vec{A} \quad\mapsto\quad j\vec{k}\cdot\hat{\vec{A}} \quad\text{and}\quad \operatorname{curl}\vec{A} \quad\mapsto\quad j\vec{k}\times\hat{\vec{A}}$$

The benefit of a formulation in reciprocal space is obvious: relatively difficult differentiation operations are mapped into relatively simple geometrical operations.

THE SPACE-TIME FOURIER TRANSFORM

There are impressive successes that have been obtained using the Fourier spatial transform in microscopic models where the constitutive relations are neither time-dependent nor space-dependent.

However, as soon as macroscopic models are used, Fourier spatial transform alone is not sufficient to simplify electrodynamics, since in reciprocal space, fields are still time-dependent. Therefore, the constitutive relations are still complicated to formulate. This could be better described in the frequency domain.

While ordinary Fourier transform or Fourier spatial transform are frequently used in separate applications, it appears as if a combination of both was not frequently applied until now. We will demonstrate that this is not justified. So let us combine these transforms in the following way. Be $f(\vec{r},t)$ a function of space and time. Then we will call

$$\breve{f}\left(j\vec{k}, j\omega\right) = \int_{\mathbb{R}^4} f(\vec{r},t)e^{-j\left(\vec{k}\cdot\vec{r}+\omega t\right)}\,dt\ d^3r$$

the space-time Fourier transform. The domain of the transformed functions will be called the *transformed space*. The inverse transform is given as

$$f(\vec{r},t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \breve{f}(j\vec{k}, j\omega) e^{+j(\vec{k}\cdot\vec{r}+\omega t)} \, d\omega \, d^3k \, .$$

All properties that are known from the ordinary Fourier transform can be transferred to the space-time transform.

Fore example, if *f* is a real-valued function, then it follows

$$\breve{f}(-j\vec{k}, -j\omega) = \left[\breve{f}(j\vec{k}, j\omega)\right]^*$$

It follows in particular for any scalar function $\psi(\vec{r},t)$:

$$\frac{d\psi}{dt} \circ\!\!-\!\!\bullet \; j\omega\breve{\psi} \quad \text{and} \quad \operatorname{grad}\psi \circ\!\!-\!\!\bullet \; j\vec{k}\,\breve{\psi}\, .$$

Similarly, we obtain for any vector $\vec{X}(\vec{r},t)$

$$\operatorname{div}\vec{X} \circ\!\!-\!\!\bullet \; j\vec{k}\cdot\breve{\vec{X}} \quad \text{and} \quad \operatorname{curl}\vec{X} \circ\!\!-\!\!\bullet \; j\vec{k}\times\breve{\vec{X}}\, .$$

The advantage of a formulation in transformed space is thus obvious: any differentiation operation with respect to time or to space-coordinates is mapped into a simple geometrical operation.

ELECTRODYNAMICS IN TRANSFORMED SPACE

We can now derive from Maxwell's equations [2]

$\operatorname{div}\vec{B}(\vec{r},t) = 0$, (law of continuity of magnetic flux)

$\operatorname{div}\vec{D}(\vec{r},t) = \rho(\vec{r},t)$, (Gauss's law)

$\operatorname{curl}\vec{E}(\vec{r},t) = -\partial\vec{B}(\vec{r},t)/\partial t$, (Faraday's law)

$\operatorname{curl}\vec{H}(\vec{r},t) = \partial\vec{D}(\vec{r},t)/\partial t + \vec{j}_C(\vec{r},t)$, (Ampère's generalized law)

the fundamental laws of electrodynamics in the transformed space as

$$j\vec{k}\cdot\breve{\vec{B}} = 0\, , \quad j\vec{k}\cdot\breve{\vec{D}} = \breve{\rho}\, ,$$
$$j\vec{k}\times\breve{\vec{E}} = -j\omega\,\breve{\vec{B}}\, , \quad j\vec{k}\times\breve{\vec{H}} = j\omega\,\breve{\vec{D}} + \breve{\vec{j}}_C\, .$$

In these equations is $\breve{\vec{B}}$ the magnetic induction, $\breve{\vec{H}}$ the magnetic field, $\breve{\vec{E}}$ the electric field, $\breve{\vec{D}}$ the electric flux density, $\breve{\rho}$ the charge density and $\breve{\vec{j}}_C$ the vector of current density.

Note that the above equations are linear algebraic equations. We will now demonstrate that many statements of electrodynamics are much easier to derive in the transformed space than in the space-time-domain.

First of all, since we see that all these fundamental equations yield statements concerning vector components that are either parallel or perpendicular to the vector $\vec{k}$, it seems to be a good idea to decompose all vector fields $\breve{\vec{X}}$ in the transformed space into one component that is parallel or *longitudinal* to $\vec{k}$ and another one that is orthogonal or *transversal* to it. We will, therefore, introduce a Cartesian coordinate system, the first axis of which is parallel to $\vec{k}$. A unit vector $\vec{e}_{-1}$ shall define the direction of that axis:

$$\vec{e}_{-1} := \vec{k}/k \quad ; \quad k := \left|\vec{k}\right|.$$

Two other unit vectors $\vec{e}_{-2}$ and $\vec{e}_{-3}$ that are perpendicular to each other and to $\vec{e}_{-1}$ are constructed in the usual way using the orthogonal basis from normal space. In order to obtain a right-handed system of basis-vectors, we require

$$\vec{e}_{-1}\times\vec{e}_{-2} = \vec{e}_{-3} \quad ; \quad \vec{e}_{-2}\times\vec{e}_{-3} = \vec{e}_{-1} \quad ; \quad \vec{e}_{-3}\times\vec{e}_{-1} = \vec{e}_{-2}\, .$$

We obtain thus the following decomposition of a vector $\breve{\vec{X}}$ into a longitudinal and a transversal component

$$\breve{\vec{X}} = \underbrace{\left(\breve{\vec{X}}\cdot\vec{e}_{-1}\right)\vec{e}_{-1}}_{\breve{\vec{X}}_{\parallel}} + \underbrace{\left\{\left(\breve{\vec{X}}\cdot\vec{e}_{-2}\right)\vec{e}_{-2} + \left(\breve{\vec{X}}\cdot\vec{e}_{-1}\right)\vec{e}_{-3}\right\}}_{\breve{\vec{X}}_{\perp}}.$$

In the special case of a medium where there are no free charges, we obtain thus $\breve{\vec{B}}_{\parallel} = 0$ and $\breve{\vec{D}}_{\parallel} = 0$. I.e.: in such a medium, magnetic induction and electric flux density can only have transversal components. Once it will have been shown to students that these fields in that medium propagate as waves parallel to the propagation vector, the above statement will gain a particular meaning: magnetic induction and electric flux density will form a pair of transversal waves in that medium! It would be much more costly to gain such a general statement in ordinary space-time.

It is also very easy to introduce the vector potential in the transformed space. Since $\vec{k}$ is orthogonal to $\breve{\vec{B}}$, we must be able to construct $\breve{\vec{B}}$ as a cross-product from $\vec{k}$ and another vector $\breve{\vec{A}}$:

$$\breve{\vec{B}} = j\vec{k}\times\breve{\vec{A}}\, .$$

From our table of correspondences, we will see at once that after inverse transform it follows

$$\vec{B} = \operatorname{curl}\vec{A}\, .$$

$\vec{A}$ is thus the well-known vector potential [2]. However, from a student's point of view, the introduction of the vector potential in transformed space is much easier to accept than by vector-analytical arguments.

From the defining equation for the vector-potential in transformed space, it will become obvious that the vector-potential is not yet determined completely, since

$$j\vec{k}\times\breve{\vec{A}} = j\vec{k}\times\left\{\breve{\vec{A}} + j\vec{k}\,\breve{U}\right\},$$

where $\breve{U}$ could be any inversely transformable function transformed space. We use the freedom in adding to the vector-potential such a function to make equations more lucid.
This proceeding is known as choosing a gauge transform. One possible gauge is a choice where then the vector-potential has only transversal components:

$$j\vec{k}\cdot\breve{\vec{A}}=0 \quad\Leftrightarrow\quad \operatorname{div}\vec{A}=0\,.$$

It is known as *Coulomb gauge* [2]. We will use it for the rest of this paper.

Eliminating the magnetic induction in Faraday's law and rearranging this equation yields

$$j\vec{k}\times\left\{\breve{\vec{E}}+j\omega\breve{\vec{A}}\right\}=0\,,$$

which means that the vector in curly brackets must be parallel to the propagation vector! Therefore it must be possible to construct it as a product of the propagation vector and a scalar $\left(j\omega, j\vec{k}\right)$-domain function $\breve{\phi}$:

$$\breve{\vec{E}}+j\omega\,\breve{\vec{A}}=-j\vec{k}\,\breve{\phi}\,.$$

Let us first consider the case where $\omega=0$, i.e. the case of time-invariant or stationary fields. In this case, the inverse space-time Fourier transform yields:

$$\vec{E}_{stat}=-\operatorname{grad}\phi\,.$$

This is the well-known result that in electrostatics the electric field can be expressed as the gradient of a scalar potential function ϕ. In all other cases, we obtain:

$$\vec{E}=-\partial\vec{A}/\partial t-\operatorname{grad}\phi$$

which is the well-known result [2] and that explains the name "vector potential".

We will now analyze the special case of a linear, homogeneous, isotropic medium. To that aim, we eliminate the electric field, electric flux density, magnetic field and magnetic induction from Gauss's and Ampère's law. Using Coulomb gauge and the well-known laws of geometry, it follows

$$k^2\,\breve{\phi}=\breve{\rho}/\hat{\varepsilon}\,,$$
$$\left(k^2-\omega^2\hat{\mu}\hat{\varepsilon}\right)\breve{\vec{A}}=\omega\hat{\mu}\hat{\varepsilon}\,\vec{k}\,\breve{\phi}+\hat{\mu}\,\breve{\vec{j}}_C\,.$$

Since in Coulomb gauge the longitudinal component of the vector-potential vanishes, it follows immediately

$$0=j\omega\hat{\varepsilon}\,k^2\,\breve{\phi}+j\vec{k}\cdot\breve{\vec{j}}_C=j\omega\breve{\rho}+j\vec{k}\cdot\breve{\vec{j}}_C \quad\Leftrightarrow\quad \operatorname{div}\vec{j}_C=-\frac{\partial\rho}{\partial t}$$

and this is the continuity equation. It remains to solve

$$k^2\,\breve{\phi}=\breve{\rho}/\hat{\varepsilon}$$
$$\left(k^2-\omega^2\hat{\mu}\hat{\varepsilon}\right)\breve{\vec{A}}_\perp=\hat{\mu}\,\breve{\vec{j}}_{C,\perp}$$

The first equation is easily solved by comparing it to the special case of a point charge in the static case. From there it follows that the inverse transform of $1/k^2$ is $1/(4\pi r)$. Using this last result and the product theorem of the Fourier transform, we obtain then that the inverse spatial Fourier transform of $\breve{\phi}$ is

$$\hat{\phi}(\vec{r})=\int_{\mathbb{R}^3}\frac{\hat{\rho}\left(\vec{r}\,',j\omega\right)}{4\pi\hat{\varepsilon}(j\omega)\left|\vec{r}-\vec{r}\,'\right|}\,d^3r'$$

In case of a frequency-independent medium it follows after ordinary inverse Fourier transform:

$$\phi(\vec{r})=\int_{\mathbb{R}^3}\frac{\rho\left(\vec{r}\,',t\right)}{4\pi\varepsilon\left|\vec{r}-\vec{r}\,'\right|}\,d^3r'\,.$$

Every lecturer who had to derive this result in ordinary space-time will admit that the method in transformed space is much easier to follow.

Let us now demonstrate how to solve the inhomogeneous wave equation for frequency-independent media. We consider

$$\left(k^2-\omega^2\hat{\mu}\hat{\varepsilon}\right)\breve{\vec{A}}_\perp=\hat{\mu}\,\breve{\vec{j}}_{C,\perp}$$

First of all, we decompose this equation into

$$\left(k-\omega\sqrt{\mu\varepsilon}\right)\left(k+\omega\sqrt{\mu\varepsilon}\right)\breve{\vec{A}}_\perp=\mu\,\breve{\vec{j}}_{C,\perp}$$

Now, we make use of the fact that $(\omega-\omega_0)\delta(\omega-\omega_0)=0$ to result in

$$(\omega-ck)(\omega+ck)\left\{\breve{\vec{A}}_\perp-2\pi\hat{\vec{A}}^{(-)}\delta(\omega+ck)-2\pi\hat{\vec{A}}^{(+)}\delta(\omega-ck)\right\}$$
$$=-c^2\mu\,\breve{\vec{j}}_{C,\perp}$$

where we have used the abbreviation $c^2=1/\varepsilon\mu$ and where vectors $\hat{\vec{A}}^{(-)}$ and $\hat{\vec{A}}^{+}$ are any transversal vectors in reciprocal space. From that it follows

$$\breve{\vec{A}}_\perp=-\frac{c^2\mu\,\breve{\vec{j}}_{C,\perp}}{\omega^2-c^2k^2}+2\pi\hat{\vec{A}}^{(-)}\delta(\omega+ck)+2\pi\hat{\vec{A}}^{(+)}\delta(\omega-ck)\,.$$

The electric field and the magnetic induction can now be found in an easy way:

$$\breve{\vec{E}}=-j\vec{k}\,\breve{\phi}-j\omega\,\breve{\vec{A}}_\perp\,,\ \breve{\vec{B}}=j\vec{k}\times\breve{\vec{A}}_\perp\,.$$

By applying the formula of inverse space-time Fourier transform, we find thus analytical formulae for the electric field and the magnetic induction in ordinary space-time.

Since the equations that we use are all linear, we can use the superposition principle to solve different parts of the problem separately. Thus let us first have a look at the solutions of the homogeneous equation for the vector potential:

$$\breve{\vec{A}}_{\perp,+}=2\pi\hat{\vec{A}}^{(+)}\delta(\omega-ck)\ \text{and}\ \breve{\vec{A}}_{\perp,-}=2\pi\hat{\vec{A}}^{(-)}\delta(\omega+ck)\,.$$

Application of the inverse ordinary Fourier transform yields

$$\hat{\vec{A}}_{\perp,+}=\hat{\vec{A}}^{(+)}e^{jckt}\ \text{and}\ \hat{\vec{A}}_{\perp,-}=\hat{\vec{A}}^{(-)}e^{-jckt}\,.$$

With $\vec{k}=k\vec{e}_{-1}$ we obtain

$$ckt=ct\,\vec{k}\cdot\vec{e}_{-1}\,.$$

Then using the shifting theorem of the Fourier transform, it follows immediately that

$$\vec{A}_{\perp,+} = \vec{A}^{(+)}\left(\vec{r} - c\,t\,\vec{e}_{-1}\right) \text{ and } \vec{A}_{\perp,-} = \vec{A}^{(-)}\left(\vec{r} + c\,t\,\vec{e}_{-1}\right)$$

where $\vec{A}^{(+)}(\vec{r})$ and $\vec{A}^{(-)}(\vec{r})$ are the inverse Fourier spatial transforms of $\hat{\vec{A}}^{(+)}\left(j\vec{k}\right)$ and $\hat{\vec{A}}^{(-)}\left(j\vec{k}\right)$. The only restriction of these vectorial functions was that they had not a component parallel to $\vec{k}$. Thus, the solutions of the homogeneous equation for the vector-potential is a superposition of any functions the graph of which propagates with velocity c parallel or anti-parallel to the vector $\vec{k}$ and this explains the name “propagation vector”. Functions with these properties are known to us as *waves*.

In the case of vanishing charge density where the scalar potential is zero, it follows that the electric field is the time derivative of the vector potential. It is thus also parallel to the propagation vector. The magnetic induction evolves from the vector-potential as the cross-product with $j\vec{k}$ in reciprocal space. It is thus orthogonal to the propagation vector as well as to the electric field vector.

We have thus proven: in the case of vanishing charge and current densities and in homogeneous, isotropic, frequency-independent material, electromagnetic fields exist as TEM-waves.

A comparable constructive proof by solution of Maxwell’s equations in ordinary space-time would have been much more difficult.
As a last example, let us consider the case

$$\breve{\vec{A}}_{\perp} = -\frac{c^2\mu\,\breve{\vec{j}}_{C,\perp}}{\omega^2 - c^2k^2}\,.$$

Using the shifting theorem and the integration theorem of the ordinary Fourier transform, we can write down an analytic expression in reciprocal space and then apply the inverse transform formula for the Fourier spatial transform to obtain an analytic expression in the time domain.

The same applies for the electric field and the magnetic induction. Thus, we have found an easier way of deriving an analytic solution of antenna problems.

SUMMARY

We have shown that the application of the space-time Fourier transform simplifies mathematics for solving electrodynamical problems considerably. Students will benefit from this simplification by having a more direct insight into physical problems instead of being occupied by understanding unloved mathematical procedures.

REFERENCES

1. Cohen-Tannoudji, C., Dupont-Roc, J., Grynberg, G., *Photons and Atoms*. New York, Chichester, Weinheim, Brisbane, Singapore, Toronto: Wiley, 11-22 (1989).
2. Jackson, J.D., *Classical Electrodynamics.* New York, Chichester, Weinheim, Brisbane, Singapore, Toronto: Wiley, (1975).